

Power Utility Maximization in Hidden Regime-Switching Markets with Default Risk

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Abstract

We consider the problem of maximizing expected utility from terminal wealth for a power investor who can allocate his wealth in a stock, a defaultable bond, and a money market account. The dynamics of these security prices are governed by geometric Brownian motions modulated by a hidden continuous time finite state Markov chain. By means of a reference probability approach to filtering in the enlarged market filtration, we reduce the partially observed stochastic control problem to a risk sensitive control problem with full observation. We separate the latter into a pre-default and a post-default dynamic optimization subproblems, and obtain two coupled Hamilton-Jacobi-Bellman equations for the optimal value functions. We obtain a complete solution to the post-default optimization subproblem, and prove a verification theorem for the solution of the pre-default optimization subproblem.

1 Introduction

Regime switching models constitute an appealing framework, stemming from their ability to capture the relevant features of asset price dynamics, which behave differently depending on the specific phase of the business cycle in place. In the stock market, Ang and Bekaert (2002a) and Ang and Bekaert (2002b) find the existence of two regimes, characterized by different volatility levels. By means of a historical analysis of the corporate bond market, Giesecke et al. (2011) identify three credit regimes, characterized by different levels of default intensity and recovery rates.

These considerations, along with their mathematical tractability, have originated a significant amount of research. In the context of continuous time utility maximization, some studies have considered observable regimes, while others have accounted for the possibility that they are not directly visible. In the case of observable regimes, Zariphopoulou (1992) considers an infinite horizon investment-consumption model where the agent can invest her wealth in a stock and risk-free bond, with borrowing and stock short-selling constraints. In a similar regime switching framework, Sotomayor and Cadenillas (2009) study the infinite horizon problem of a risk averse investor maximizing regime dependent utility from terminal wealth and consumption. Elliott and Siu (2010) study the risk minimization problem as a stochastic differential game within a regime-switching framework. A different branch of literature has considered the case when regimes are hidden and need to be estimated from publicly available market information. Nagai and Runggaldier (2008) consider a finite horizon portfolio optimization problem, where a power investor allocates his wealth across money market account and stocks, whose price dynamics follow a diffusion process modulated by a hidden finite-state Markov process. Tamura and Watanabe (2011) extend the analysis to the case when the time horizon is infinite. Siu (2011) includes the possibility of investing in inflation-linked bonds in an economy modulated by hidden regimes. Using a similar framework, Elliott and Siu (2011) study the optimal investment problem of an insurer when the model uncertainty is governed by a hidden Markov chain. In the context of optimal inventory management, Cadenillas et al. (2012) model the demand of an item using an arithmetic Brownian motion, whose drift is modulated by a two states Markov chain. They recover the optimal production policy, both in the case of observable

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and hidden regimes, using the technique of “completing squares”. Sass and Haussmann (2004) consider a multi-stock market model, with stochastic interest rates and drift modulated by a hidden Markov chain. Combining appropriate Malliavin calculus and filtering results from hidden Markov models, they derive explicit representations of the optimal strategies. In a series of two papers, Fujimoto et al. (2012a) and Fujimoto et al. (2012b) consider a regime switching framework where logarithmic and power investors optimize their terminal utility using stock prices received randomly according to a Cox process. The random arrival process leads to incomplete information on the underlying Markov chain.

The literature surveyed above has considered markets consisting of securities carrying market, but not default, risk. However, the recent credit events, combined with empirical evidence showing that corporate bonds markets have a similar capitalization to all publicly traded companies in the United States, stress the importance of including defaultable securities in portfolio decision problems. The first attempt in this direction was made by Capponi and Figueroa-López (2012), who consider a portfolio optimization framework where investors can trade both a stock and a defaultable security. Using the HJB approach, they recover the optimal investment strategies as the unique solution to a coupled system of partial differential equations, and characterize the directionality of the bond investment strategy by means of a relation between historical and risk-neutral default risk components. However, they assume regimes to be observable and, as such, the optimal investment strategy is highly sensitive to the regime in place.

This paper considers the case where regimes are hidden, and a power investor has to decide on his optimal allocation policy using only the observed market prices. This improves upon the realism of the model in Capponi and Figueroa-López (2012), given that in several circumstances market regimes such as inflation and recession, or credit regimes characterized by high or low credit spreads, are typically unobserved to investors or discovered with significant time lag. Moreover, the hidden regime feature requires a completely different analysis, and leads us to solving a partially observed stochastic control problem, where regime information must be inferred from an enlarged market filtration. The latter is composed of *both* a reference filtration generated by the observable security prices, *and* a credit filtration tracking the occurrence of the default event. To the best of our knowledge, ours represents the first study in this direction.

We next describe our main contributions. We consider a portfolio optimization problem for a power investor in a context of partial information and the possibility of default. This advances earlier literature, which has so far considered either one or the other aspect, but never both simultaneously. We construct an equivalent fully observed risk-sensitive control problem, where the new state is given by the normalized regime filtered probabilities. Although we follow a similar approach to Nagai and Runggaldier (2008) for our control problem, there are important differences which require a nontrivial analysis. Firstly, while the filter probabilities in Nagai and Runggaldier (2008) are directly given by the Wonham filter (cf. Wonham (1965)), in our case the dynamics are not of diffusive type, and are obtained by an extension of previous filtering results dealing with the enlarged credit filtration. We follow the reference probability approach and construct a probability measure such that the underlying chain becomes independent of both the market price process *and* the default indicator event. We remark that Frey and Runggaldier (2010), Section 4.1, also consider filter equations for finite-state Markov chains in the presence of multiple default events. However, they provide the dynamics of the unnormalized filter probabilities using a Zakai-type SDE, and then construct an algorithm to compute the filter probabilities. In the context of pricing credit derivatives, Frey and Schmidt (2012) study a similar model to Frey and Runggaldier (2010), but provide the dynamics of the filter probabilities using the innovation approach to filtering. We obtain the filter probability dynamics using the reference probability method, as our methodology requires to reformulate the control problem under such a reference measure. We then use the latter probabilities to obtain the Hamilton-Jacobi-Bellman (HJB) for the dynamical optimization problem, which we separate it into a coupled pre-default and post-default dynamical optimization subproblems. This is done using the projected filter process, as opposed to Nagai and Runggaldier (2008) who consider the degenerate filter process in their analysis. Because of the default event, the HJB-PDE satisfied by the pre-default value function is nonlinear. Using the Hopf-Cole transformation, we obtain a nonlinear parabolic PDE which degenerates at the boundary of the domain, and prove a novel verification theorem for its solution. By contrast, the HJB-PDE satisfied by the post-default value function can be linearized using a similar transformation to the one adopted by Nagai and Runggaldier (2008), and a unique

classical solution solving the post-default optimization subproblem can be guaranteed.

The rest of the paper is organized as follows. Section 2 defines the market model. Section 3 sets up the utility maximization problem. Section 4 derives the HJB equations corresponding to the risk sensitive control problem. Section 5 analyzes the solutions of the HJB-PDE equations. Section 6 summarizes our main conclusions. Finally, two appendices present the main proofs of the paper.

2 The Market Model

Assume $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ is a complete filtered probability space, where \mathbb{P} is the real world probability measure (also called historical probability), $\mathbb{G} := (\mathcal{G}_t)_t$ is an enlarged filtration given by $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$, where $(\mathcal{H}_t)_t$ is a filtration to be introduced below. Here, $\mathbb{F} := (\mathcal{F}_t)_t$ is a suitable filtration supporting a two dimensional Brownian motion $W_t = (W_t^{(1)}, W_t^{(2)})$. We also assume that the *hidden* states of the economy are modeled by a continuous-time Markov process $\{X_t\}$ defined on $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ with a finite state space identified by the set of unit vectors $\{e_1, e_2, \dots, e_N\}$, where $e_i = (0, \dots, 1, \dots, 0)' \in \mathbb{R}^N$ and $'$ denotes the transpose. The following semi-martingale representation is well-known (cf. Elliott et al. (1994)):

$$X_t = X_0 + \int_0^t A(s)' X_s ds + \varphi(t), \quad (1)$$

where $\varphi(t) = (\varphi_1(t), \dots, \varphi_N(t))'$ is a \mathbb{R}^N -valued martingale process under \mathbb{P} , and $A(t) := [\varpi_{i,j}(t)]_{i,j=1,\dots,N}$ is the so-called generator of the Markov process. Specifically, denoting $p_{i,j}(t, s) := \mathbb{P}(X_s = e_j | X_t = e_i)$, for $s \geq t$, and $\delta_{i,j} = \mathbf{1}_{i=j}$, we have that

$$\varpi_{i,j}(t) = \lim_{h \rightarrow 0} \frac{p_{i,j}(t, t+h) - \delta_{i,j}}{h};$$

cf. Bielecki and Rutkowski (2001). In particular, $\varpi_{i,i}(t) = -\sum_{j \neq i} \varpi_{i,j}(t)$. We denote by $p_0 = (p_0^1, \dots, p_0^N)$ the initial distribution on the Markov chain and, throughout the paper, assume that $p_0^i > 0$. We consider a frictionless financial market consisting of three instruments: a risk-free bank account, a defaultable bond, and a stock. The dynamics of each of the following instruments will depend on the underlying states of the economy as follows:

Risk-free bank account. The instantaneous market interest rate is assumed to be constant. The dynamics of the price process $\{B_t\}$, which describes the risk-free bank account, is given by

$$dB_t = rB_t dt, \quad B_0 = 1. \quad (2)$$

Stock price. We assume that the appreciation rate $\{\mu_t\}$ of the stock also depends on the economy regime X_t in the following way:

$$\mu_t := \mu(t, X_t) := \langle \mu, X_t \rangle,$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_N)'$ is a vector denoting the values of drift which can be taken depending on the different economic regimes. The volatility σ is assumed to be constant. Hence, under the historical measure, the stock dynamics is given by

$$dS_t = \mu_t S_t dt + \sigma S_t dW_t^{(1)}, \quad S_0 = s,$$

where $\{W_t^{(1)}\}$ is a standard Brownian motion on $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$.

Risky bond price. Before defining the bond price, we need to introduce a default process. Let τ be a nonnegative random variable, defined on $(\Omega, \mathcal{G}, \mathbb{P})$, representing the default time of the counterparty selling the bond. Let $\mathcal{H}_t = \sigma(H(u) : u \leq t)$ be the filtration generated by the *default process* $H(t) := H_t := \mathbf{1}_{\tau \leq t}$, after completion and regularization on the right (see Belanger et al. (2004)). We use the canonical construction of the default time τ , in terms of a given hazard process $\{h_t\}_{t \geq 0}$, which will be specified later. For future reference, we now give the details of

the construction of the random time τ . We assume the existence of an exponential random variable χ defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, independent of the process $(X_t)_t$. We define τ by setting

$$\tau := \inf \left\{ t \in \mathbb{R}^+ : \int_0^t h_u du \geq \chi \right\}. \quad (3)$$

It can be proven that $(h_t)_t$ is the (\mathbb{F}, \mathbb{G}) -hazard rate of τ (see Bielecki and Rutkowski (2001), Section 6.5 for details). That is, $(h_t)_t$ is such that

$$\xi_t := H(t) - \int_0^t \bar{H}(u^-) h_u du = H(t) - \int_0^{t \wedge \tau} h_u du, \quad (4)$$

is a \mathbb{G} -martingale under \mathbb{P} , where $\bar{H}(u) := \bar{H}_u := 1 - H(u)$ and $\bar{H}(u^-) := \lim_{s \uparrow u} \bar{H}(s) = \mathbf{1}_{\tau \geq u}$. Intuitively, Eq. (4) says that the single jump process needs to be compensated for default, prior to the occurrence of the event. As with the appreciation rate, we assume that the process h is driven by the hidden Markov chain as follows:

$$h_t := \langle h, X_t \rangle,$$

where $h = (h_1, \dots, h_N)'$ denotes the possible values that the default rate process can take depending on the economic regime in place.

Following Frey and Runggaldier (2011), we assume that the yield of the defaultable bond is only observed up to noise by market investors. We postulate that, at time $t_k = k\Delta$, the observed yield z_{t_k} is such that

$$z_{t_k} - z_{t_{k-1}} = a(t_k, X_{t_k}, H_{t_k})\Delta + \epsilon_k,$$

where $a(t, X_t, H_t)$ denotes the actual yield and $(\epsilon_k)_k$ is an i.i.d. sequence, independent of X and $W^{(1)}$, which models transaction costs such as bid/ask spreads and other market frictions. In light of Donsker's invariance principle, as in Frey and Runggaldier (2011), the observed yield process is expected to converge to

$$z_t = \int_0^t a(s, X_s, H_s) ds + v W_t^{(2)}, \quad (5)$$

when $\Delta \rightarrow 0$, where $W_t^{(2)}$ is a Brownian motion independent of X and $W^{(1)}$. The previous heuristics serve as motivation to define the observed bond price as $P_t = P_0 e^{z_t}$, with $(z_t)_t$ given by the dynamics (5). Equivalently, hereafter the dynamics of the observed bond price $(P_t)_t$ is assumed to be given by

$$\frac{dP_t}{P_t} = \left(a(t, X_t, H_t) + \frac{1}{2} v^2 \right) dt + v dW_t^{(2)},$$

before default. Note that, if the actual yields were observed without noise (hence, $v = 0$), we would recover the standard dynamics of a bond with continuously compounded yield $a(t, X_t, H_t)$. Throughout the paper, we denote the maturity of the bond by $\mathcal{T} > 0$.

We define two subfiltrations of \mathbb{G} , namely, the market filtration $\mathbb{G}^I := (\mathcal{G}_t^I)_t$ where

$$\mathcal{G}_t^I := \mathcal{F}_t^I \vee \mathcal{H}_t, \quad \mathcal{F}_t^I := \sigma(S_u, P_u; u \leq t),$$

and the filtration $\mathbb{F}^X := (\mathcal{F}_t^X)_t$, generated by the Markov chain $(X_t)_t$:

$$\mathcal{F}_t^X = \sigma(X_u; u \leq t).$$

Therefore, we may also write $\mathcal{G}_t = \mathcal{F}_t^X \vee \mathcal{G}_t^I$. From this, it is evident that, while $(X_t)_t$ is $(\mathcal{G}_t)_t$ adapted, it is not $(\mathcal{G}_t^I)_t$ adapted.

3 The Utility Maximization Problem

We consider an investor who wants to maximize her expected final utility during a trading period $[0, T]$, with $T < \mathcal{T}$, by dynamically allocating her financial wealth into (1) the risk-free bank account, (2) the stock, and (3) the defaultable \mathcal{T} -bond, as defined in the previous section. Let us denote by ν_t^B the number of shares of the risk-free bank account that the investor buys ($\nu_t^B > 0$) or sells ($\nu_t^B < 0$) at time t . Similarly, ν_t^S and ν_t^P denote the investor's portfolio positions in the stock and risky bond at time t , respectively. The process $(\nu_t^B, \nu_t^S, \nu_t^P)$ is called a *portfolio process*. We denote $V_t(\nu)$ the wealth of the portfolio process $\nu = (\nu^B, \nu^S, \nu^P)$ at time t , i.e.

$$V_t(\nu) = \nu_t^B B_t + \nu_t^S S_t + \nu_t^P P_t.$$

We require the processes ν_t^B, ν_t^S , and ν_t^P to be \mathcal{G}^I -predictable. The investor does not have intermediate consumption nor capital income to support her trading of financial assets and, hence, we also assume the following self-financing condition:

$$dV_t = \nu_t^B dB_t + \nu_t^S dS_t + \nu_t^P dP_t.$$

Let

$$\pi_t^B := \frac{\nu_t^B B_t}{V_{t-}(\nu)}, \quad \pi_t^S := \frac{\nu_t^S S_t}{V_{t-}(\nu)}, \quad \pi_t^P = \frac{\nu_t^P P_t}{V_{t-}(\nu)} \mathbf{1}_{\tau > t}, \quad (6)$$

if $V_{t-}(\nu) > 0$, while $\pi_t^B = \pi_t^P = \pi_t^S = 0$ when $V_{t-}(\nu) = 0$. The vector $\pi_t := (\pi_t^B, \pi_t^S, \pi_t^P)$, called a *trading strategy*, represents the corresponding fractions of wealth invested in each asset at time t . Note that if $\pi := (\pi_t)_t$ is admissible (the precise definition will be given later), then the dynamics of the resulting wealth process V^π can be written as

$$dV_t^\pi = V_{t-}^\pi \left\{ \pi_t^B \frac{dB_t}{B_t} + \pi_t^S \frac{dS_t}{S_t} + \pi_t^P \frac{dP_t}{P_t} \right\}, \quad (7)$$

under the convention that $0/0 = 0$. This convention is needed to deal with the case when default has occurred ($t > \tau$), so that $P_t = 0$ and we fix $\pi_t^P = 0$. We have the following dynamics of the wealth process

$$\frac{dV_t}{V_t} = rdt + \pi_t^S (\mu_t - r) dt + \pi_t^S \sigma dW_t^{(1)} + \pi_t^P \left(a(t, X_t, H_t) + \frac{v^2}{2} - r \right) dt + \pi_t^P v dW_t^{(2)}, \quad V_0 = v,$$

for a given initial budget $v \in (0, \infty)$. The objective is to choose π so to maximize the expected terminal utility

$$J(v, \pi, T) := \frac{1}{\gamma} \mathbb{E}[V_T^\gamma], \quad (8)$$

for a given fixed value of γ in $(0, 1)$. From here on, we use $\pi := (\pi_t^S, \pi_t^P)'$ to denote the investment strategy, only consisting of stock and defaultable bond due to the self-financing property.

In the sequel, it would be useful to have an explicit formula for V_t^γ . By Itô's formula and Eq. (7), we readily obtain that

$$\begin{aligned} dV_t^\gamma &= \gamma V_t^\gamma \left[rdt + \pi_t^S (\mu_t - r) dt + \pi_t^S \sigma dW_t^{(1)} + \pi_t^P \left(a(t, X_t, H_t) + \frac{v^2}{2} - r \right) dt + \pi_t^P v dW_t^{(2)} \right] \\ &\quad + \frac{1}{2} \gamma (\gamma - 1) V_t^\gamma [(\pi_t^S)^2 \sigma^2 dt + (\pi_t^P)^2 v^2 dt]. \end{aligned}$$

Next, define

$$\Sigma_Y = \begin{pmatrix} \sigma & 0 \\ 0 & v \end{pmatrix},$$

and recall that $W_t := (W_t^{(1)}, W_t^{(2)})'$ and $\pi_t = (\pi_t^S, \pi_t^P)'$. Then, we may rewrite the above SDE as

$$dV_t^\gamma = V_t^\gamma [-\gamma \eta(t, X_t, H_t, \pi_t) dt + \gamma \pi_t' \Sigma_Y dW_t], \quad (9)$$

where

$$\eta(t, X_t, H_t, \pi_t) = -r + \pi_t^S(r - \mu_t) + \pi_t^P \left(r - \frac{1}{2}v^2 - a(t, X_t, H_t) \right) + \frac{1-\gamma}{2} \pi_t' \Sigma_Y' \Sigma_Y \pi_t. \quad (10)$$

It is then clear that the solution to the stochastic differential equation (9) with initial condition $V_0 = v$ is given by

$$V_t^\gamma = v^\gamma \exp \left(\gamma \int_0^t \pi_s' \Sigma_Y dW_s - \gamma \int_0^t \eta(s, X_s, H_s, \pi_s) ds - \frac{\gamma^2}{2} \int_0^T \pi_s' \Sigma_Y \Sigma_Y' \pi_s ds \right).$$

3.1 Change of Probability Measure

We follow the reference probability approach to filtering, and develop a change of measure under which the underlying chain $(X_t)_t$ becomes independent of the investor filtration \mathbb{G}^I . First, we introduce further notation and terminology. Given two semimartingales L and M , we denote by $[L]$ and $[L, M]$ the quadratic variation of L and the quadratic covariation of L and M , respectively. We also denote the stochastic exponential of L by $\mathcal{E}(L)$. If L is of the form $L_t = \int_0^t \theta_s' dY_s$, where Y_s is a \mathbb{R}^d -valued continuous Itô process, and $(\theta_t)_t$ is \mathbb{G} predictable, then

$$\mathcal{E}_t(L) = \exp \left(\int_0^t \theta_u' dY_u - \frac{1}{2} \int_0^t \theta_u' \theta_u d[Y]_u \right). \quad (11)$$

If Z is of the form $Z_t = \int_0^t \kappa_s d\xi_s$, where ξ_s has been defined in (4), and κ_s is \mathbb{G} -predictable, with $\kappa > -1$, then

$$\mathcal{E}_t(Z) = \exp \left(\int_0^t \log(1 + \kappa_s) dH_s - \int_0^{t \wedge \tau} \kappa_s h_s ds \right). \quad (12)$$

It is well known (see Bielecki and Rutkowski (2001), Section 3.4) that $R_t := \mathcal{E}_t(L)\mathcal{E}_t(Z)$ follows the SDE

$$R_t = 1 + \int_{(0,t]} R_{s-} (\theta_s' dY_s + \kappa_s d\xi_s). \quad (13)$$

We now proceed to introduce a new measure $\hat{\mathbb{P}}$ on (Ω, \mathbb{G}) , under which the hidden chain $(X_t)_t$ becomes independent of $(\mathcal{G}_t^I)_{t \geq 0}$. Such a measure is defined in terms of its density process $(\rho_t)_t$ as follows:

$$\rho_t := \frac{d\hat{\mathbb{P}}}{d\mathbb{P}} \Big|_{\mathcal{G}_t} = \mathcal{E}_t \left(\int_0^\cdot -\vartheta(s, X_s, H_s)' (\Sigma_Y \Sigma_Y')^{-1} \Sigma_Y dW_u \right) \mathcal{E}_t \left(\int_0^\cdot \frac{1 - h_{s-}}{h_{s-}} d\xi_s \right) =: \rho_t^{(1)} \rho_t^{(2)}, \quad (14)$$

where

$$\vartheta(t, X_t, H_t) := \left[\mu_t - \frac{\sigma^2}{2}, a(t, X_t, H_t) \right]' = \left[\langle \mu, X_t \rangle - \frac{\sigma^2}{2}, a(t, X_t, H_t) \right]'. \quad (15)$$

In particular, using Eqs. (11) and (12), $\rho_t^{(1)}$ and $\rho_t^{(2)}$ above are given by

$$\begin{aligned} \rho_t^{(1)} &= \exp \left(- \int_0^t \vartheta(s, X_s, H_s)' (\Sigma_Y \Sigma_Y')^{-1} \Sigma_Y dW_s - \frac{1}{2} \int_0^T \vartheta' (\Sigma_Y \Sigma_Y')^{-1} \vartheta(s, X_s, H_s) ds \right), \\ \rho_t^{(2)} &= \exp \left(- \int_0^t \log(h_{u-}) dH_u - \int_0^{t \wedge \tau} (1 - h_u) du \right) = h_{\tau-}^{-1\{\tau \leq t\}} \exp \left(- \int_0^{t \wedge \tau} (1 - h_u) du \right). \end{aligned}$$

Moreover, from Eq. (13), ρ_t admits the following representation

$$\rho_t = 1 + \int_{[0,t]} \rho_{s-} \left(-\vartheta(s, X_s, H_s)' (\Sigma_Y \Sigma_Y')^{-1} \Sigma_Y dW_s + \frac{1 - h_{s-}}{h_{s-}} d\xi_s \right).$$

Under such a measure, by Girsanov theorem (see also Bielecki and Rutkowski (2001), Section 5.3), we have that

$$\hat{W}_t = W_t + \int_0^t \Sigma_Y' (\Sigma_Y \Sigma_Y')^{-1} \vartheta(s, X_s, H_s) ds$$

is a Brownian motion, and

$$\hat{\xi}_t = \xi_t - \int_0^{t \wedge \tau} (1 - h_u) du = H_t - \int_0^{t \wedge \tau} du = H_t - \int_0^t \bar{H}(u^-) du \quad (16)$$

is a \mathbb{G}^I -martingale under $\hat{\mathbb{P}}$. The inverse density process,

$$U_t := \frac{d\mathbb{P}}{d\hat{\mathbb{P}}} \Big|_{\mathcal{G}_t},$$

can be written as $U_t = U_t^{(1)} U_t^{(2)}$, where

$$\begin{aligned} U_t^{(1)} &:= \exp \left(\int_0^t \vartheta(s, X_s, H_s)' (\Sigma_Y \Sigma_Y')^{-1} dW_s + \frac{1}{2} \int_0^t \vartheta' (\Sigma_Y \Sigma_Y')^{-1} \vartheta(s, X_s, H_s) ds \right) \\ U_t^{(2)} &:= \mathcal{E}_t \left(\int_0^\cdot (h_{s-} - 1) d\hat{\xi}_s \right) = h_{\tau-}^{\mathbf{1}_{\{\tau \leq t\}}} \exp \left(\int_0^{t \wedge \tau} (1 - h_u) du \right), \end{aligned}$$

with the last equality following from Eq. (12).

Using the previous probability measure $\hat{\mathbb{P}}$, we obtain that Eq. (8) may be rewritten as

$$\begin{aligned} \frac{1}{\gamma} \mathbb{E} [V_T^\gamma] &= \frac{v^\gamma}{\gamma} \mathbb{E}^{\hat{\mathbb{P}}} \left[e^{-\gamma \int_0^T \eta(s, X_s, H_s, \pi_s) ds + \gamma \int_0^T \pi_s' \Sigma_Y dW_s - \frac{\gamma^2}{2} \int_0^T \pi_s' \Sigma_Y \Sigma_Y' \pi_s ds} U_T \right] \\ &= \frac{v^\gamma}{\gamma} \mathbb{E}^{\hat{\mathbb{P}}} [L_T], \end{aligned} \quad (17)$$

where

$$\begin{aligned} L_t &= \mathcal{E}_t \left(\int_0^\cdot Q'(s, X_s, H_s, \pi_s) dY_s \right) U_t^{(2)} \exp \left(-\gamma \int_0^t \eta(s, X_s, H_s, \pi_s) ds \right), \\ Q(s, X_s, H_s, \pi_s) &= (\Sigma_Y \Sigma_Y')^{-1} \vartheta(s, X_s, H_s) + \gamma \pi_s, \end{aligned} \quad (18)$$

and the stochastic exponential is given by

$$\mathcal{E}_t \left(\int_0^\cdot Q'(s, X_s, H_s, \pi_s) dY_s \right) = \exp \left(\int_0^t Q'(s, X_s, H_s, \pi_s) dY_s - \frac{1}{2} \int_0^t Q' \Sigma_Y \Sigma_Y' Q(s, X_s, H_s, \pi_s) ds \right).$$

Next, define

$$q_t^i = \mathbb{E}^{\hat{\mathbb{P}}} \left[L_T \mathbf{1}_{\{X_t = e_i\}} \middle| \mathcal{G}_t^I \right], \quad q_t = (q_t^1, q_t^2, \dots, q_t^N).$$

Then, we have

$$\begin{aligned} \mathbb{E} [V_T^\gamma] &= v^\gamma \mathbb{E}^{\hat{\mathbb{P}}} \left[\mathbb{E}^{\hat{\mathbb{P}}} [L_T | \mathcal{G}_T^I] \right] = v^\gamma \sum_{i=1}^N \mathbb{E}^{\hat{\mathbb{P}}} \left[\mathbb{E}^{\hat{\mathbb{P}}} [L_T \mathbf{1}_{\{X_T = e_i\}} | \mathcal{G}_T^I] \right] \\ &= v^\gamma \mathbb{E}^{\hat{\mathbb{P}}} \left[\sum_{i=1}^N q_T^i \right]. \end{aligned}$$

3.2 The Filter Probabilities

The goal of this section is to derive the dynamics of the filter probabilities of the Markov chain X_t conditioned on the investor filtration \mathcal{G}_t^I . We first derive the dynamics of $\{q_t^i\}$, which play the role of unnormalized filter probabilities. We then obtain the corresponding dynamics for the normalized filter probabilities via Itô's rule.

We start with the following lemma, whose proof is reported in Appendix A. Below, we make use of the functions η and Q defined in Eq. (10) and (18), respectively.

Lemma 3.1. *The dynamics of $(q_t^i)_{t \geq 0}$, $i = 1, \dots, N$, under the measure $\hat{\mathbb{P}}$, is given by the following system of stochastic differential equations (SDE):*

$$\begin{aligned} dq_t^i &= \sum_{\ell=1}^N \varpi_{\ell,i}(t) q_t^\ell dt + q_t^i Q'(t, e_i, H_t, \pi_t) \Sigma_Y d\hat{W}_t + q_{t-}^i (h_i - 1) d\hat{\xi}_t - \gamma \eta(t, e_i, H_t, \pi_t) q_t^i dt, \\ q_0^i &= p_0^i. \end{aligned} \quad (19)$$

Next, we introduce some notation. Let

$$M_t := \sum_{i=1}^N q_t^i, \quad (20)$$

and let p_t^i be the filter probability that the regime X_t is e_i at time t , conditional on the filtration \mathcal{G}_t^I ; that is,

$$p_t^i := \mathbb{P}(X_t = e_i | \mathcal{G}_t^I) = \frac{q_t^i}{M_t}. \quad (21)$$

We will also make use of the following short-hand notation :

$$\hat{f}_t := \mathbb{E}^{\mathbb{P}}[f(X_t) | \mathcal{G}_t^I] = \sum_{i=1}^N f(e_i) p_t^i, \quad g(t, p_t, z) = \sum_{i=1}^N g(t, e_i, z) p_t^i, \quad u(t, p_t, z, \pi) = \sum_{i=1}^N u(t, e_i, z, \pi) p_t^i, \quad (22)$$

where $f : \{e_1, \dots, e_N\} \rightarrow \mathbb{R}$, $g : \mathbb{R}_+ \times \{e_1, \dots, e_N\} \times \{0, 1\} \rightarrow \mathbb{R}$, and $u : \mathbb{R}_+ \times \{e_1, \dots, e_N\} \times \{0, 1\} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are given functions. Let us also define the simplex

$$\Delta_{N-1} = \{(d^1, d^2, \dots, d^N) : d^1 + d^2 + \dots + d^N = 1, 0 \leq d^i \leq 1, i = 1, \dots, N\}.$$

We have the following result, whose proof is reported in Appendix A.

Proposition 3.2. *The normalized filter probabilities are governed by the SDE*

$$\begin{aligned} dp_t^i &= \sum_{\ell=1}^N \varpi_{\ell,i}(t) p_t^\ell dt + p_t^i (\vartheta'(t, e_i, H_t) - \vartheta'(t, p_t, H_t)) (\Sigma_Y \Sigma_Y')^{-1} (\Sigma_Y dW_t) \\ &\quad + p_{t-}^i \frac{h_i - \hat{h}_{t-}}{\hat{h}_{t-}} \left(\Delta H_t - \hat{h}_{t-} \bar{H}(t^-) dt \right), \end{aligned} \quad (23)$$

where, in agreement with the notation introduced in Eq. (22), we had set

$$\hat{h}_t := \sum_{i=1}^N h_i p_t^i, \quad \vartheta(t, p_t, H_t) := \sum_{i=1}^N \vartheta(t, e_i, H_t) p_t^i.$$

We conclude with a technical result, which will be needed in the verification theorem proven in Section 5. Its proof is reported in Appendix A.

Lemma 3.3. *For any $T > 0$ and $i = 1, \dots, N$, it holds that*

$$\mathbb{P}(p_t^i > 0, \text{ for all } t \in [0, T]) = 1.$$

3.3 The Risk-Sensitive Control Problem

We now proceed to show how the original partially observed control problem is reduced to a risk sensitive control problem with full information. Define

$$\hat{L}_t = \mathcal{E}_t \left(\int_0^\cdot Q'(s, p_s, H_s, \pi_s) dY_s \right) \mathcal{E}_t \left(\int_0^\cdot (\hat{h}_{s-} - 1) d\hat{\xi}_s \right) e^{-\gamma \int_0^t \eta(s, p_s, H_s, \pi_s) ds},$$

where, using the shorthand notation (22), we set $\eta(t, p_t, H_t, \pi_t) = \sum_{i=1}^N \eta(t, e_i, H_t, \pi_t) p_t^i$. Then, we have the following equivalent formulation of the objective function $J(v, \pi, T)$ given in (8). The proof is reported in Appendix A.

Proposition 3.4. *It holds that*

$$J(v, \pi, T) = \frac{v^\gamma}{\gamma} \mathbb{E}^{\tilde{\mathbb{P}}} [\hat{L}_T]. \quad (24)$$

Using Eq. (24), we now demonstrate how the stochastic control problem of maximizing $J(v, \pi, T)$ can be reduced to a finite dimensional risk-sensitive stochastic control problem. This is obtained building on the approach of Nagai and Runggaldier (2008), who do not consider the defaultable security. To start with, define the measure $\tilde{\mathbb{P}}$ such that

$$\zeta_t := \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \Big|_{\mathcal{G}_t^I} = \mathcal{E}_t \left(\int_0^\cdot Q'(s, p_s, H_s, \pi_s) dY_s \right) \mathcal{E}_t \left(\int_0^\cdot (\hat{h}_{s-} - 1) d\hat{\xi}_s \right), \quad (25)$$

where, by Girsanov theorem, we have that

$$\tilde{W}_t = \hat{W}_t - \int_0^t \Sigma_Y' Q(s, p_s, H_s, \pi_s) ds$$

is a \mathbb{G}^I -Brownian motion under $\tilde{\mathbb{P}}$. Further, again by Girsanov, we have that

$$\tilde{\xi}_t = \hat{\xi}_t - \int_0^{t \wedge \tau} (\hat{h}_s - 1) ds = H_t - \int_0^{t \wedge \tau} \hat{h}_s ds = H_t - \int_0^t \hat{h}_s \bar{H}(s^-) ds \quad (26)$$

is a \mathbb{G}^I martingale under $\tilde{\mathbb{P}}$. It can then be verified that the dynamics of p_t^i in Eq. (23) may be rewritten as

$$\begin{aligned} dp_t^i &= p_t^i (\vartheta(t, e_i, H_t)' - \vartheta(t, p_t, H_t)') (\Sigma_Y \Sigma_Y')^{-1} \Sigma_Y d\tilde{W}_t \\ &+ \left(\sum_{\ell=1}^N \varpi_{\ell,i}(t) p_t^\ell + \gamma p_t^i (\vartheta(t, e_i, H_t)' - \vartheta(t, p_t, H_t)') \pi_t \right) dt + p_{t-}^i \frac{h_i - \hat{h}_{t-}}{\hat{h}_{t-}} d\tilde{\xi}_t. \end{aligned} \quad (27)$$

It follows immediately that

$$J(v, \pi, T) = \frac{v^\gamma}{\gamma} \mathbb{E}^{\tilde{\mathbb{P}}} [\hat{L}_T \zeta_T^{-1}] = \frac{v^\gamma}{\gamma} \mathbb{E}^{\tilde{\mathbb{P}}} \left[e^{-\gamma \int_0^T \eta(s, p_s, H_s, \pi_s) ds} \right], \quad (28)$$

thus showing that the original problem $J(v, \pi, T)$ is reduced to a risk sensitive control problem, where maximization is done across suitable strategies $(\pi_t)_t$ such that

$$\mathbb{E}^{\tilde{\mathbb{P}}} [\zeta_T] = \mathbb{E} [\rho_T \zeta_T] = 1, \quad (29)$$

and subject to the control process $(p_t)_t$ lying on the simplex Δ_{N-1} and following the SDE given by Eq. (27), under $\tilde{\mathbb{P}}$. We shall specify later on the precise class of trading strategies π on which the portfolio optimization problem is defined. The next section provides the HJB equation for this problem.

4 HJB formulation

This section is devoted to formulating the HJB equation. Given that the process $p = (p^1, \dots, p^N)$ is degenerate in \mathbb{R}^N , we consider the projected $N - 1$ dimensional process

$$\tilde{p}_s := (\tilde{p}_s^1, \dots, \tilde{p}_s^{N-1})' := (p_s^1, \dots, p_s^{N-1})',$$

as opposed to the filtering process. Next, we rewrite the problem (28) in terms of the above process, which now lies in the space

$$\tilde{\Delta}_{N-1} = \{ (d^1, \dots, d^{N-1})' : d^1 + \dots + d^{N-1} \leq 1, d^i \geq 0 \}.$$

Let us start with some notation needed to write the SDE of \tilde{p} in matrix form. First, similarly to (22), given a function $f : \mathbb{R}_+ \times \{e_1, \dots, e_N\} \times \{0, 1\} \times \mathbb{R}^2 \rightarrow \mathbb{R}$, we define the function $\tilde{f} : \mathbb{R}_+ \times \tilde{\Delta}_{N-1} \times \{0, 1\} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$\tilde{f}(t, d, z, \pi) = f(t, e_N, z, \pi) + \sum_{i=1}^{N-1} [f(t, e_i, z, \pi) - f(t, e_N, z, \pi)] d^i, \quad \text{for } d = (d^1, \dots, d^{N-1})', \quad (30)$$

and, given a function $g : \mathbb{R}_+ \times \{e_1, \dots, e_N\} \times \{0, 1\} \rightarrow \mathbb{R}$, we define

$$g(t, d, z) = \sum_{i=1}^N g(t, e_i, z) + \sum_{i=1}^{N-1} [g(t, e_i, z) - g(t, e_N, z)] d^i, \quad \text{for } d = (d^1, \dots, d^{N-1})'. \quad (31)$$

Similarly, with certain abuse of notation, given an N -dimensional vector $f = (f_1, \dots, f_N)'$, we introduce the function $\tilde{f} : \tilde{\Delta}_{N-1} \rightarrow \mathbb{R}$ defined as

$$\tilde{f}(d) = f_N + \sum_{i=1}^{N-1} [f_i - f_N] d^i, \quad d = (d^1, \dots, d^{N-1})',$$

and the projection of f on the first $N-1$ coordinates as $f^\perp := (f_1, \dots, f_{N-1})'$. Throughout, we use $D(\mathbf{b})$ to denote the diagonal matrix, whose i -th diagonal element is the i^{th} component of a vector \mathbf{b} . Further, let $\vartheta_{1:N-1}(t, H_t)$ and $\tilde{\beta}(t, \tilde{p}_t)$ be the $2 \times (N-1)$ matrix and $(N-1) \times 1$ vector defined by

$$\begin{aligned} \vartheta_{1:N-1}(t, H_t) &= (\vartheta(t, e_1, H_t), \dots, \vartheta(t, e_{N-1}, H_t)), \\ \tilde{\beta}(t, \tilde{p}_t) &= \left(\varpi_{N,1}(t) + \sum_{\ell=1}^{N-1} [\varpi_{\ell,1}(t) - \varpi_{N,1}(t)] \tilde{p}_t^\ell, \dots, \varpi_{N,N-1}(t) + \sum_{\ell=1}^{N-1} [\varpi_{\ell,N-1}(t) - \varpi_{N,N-1}(t)] \tilde{p}_t^\ell \right)'. \end{aligned}$$

We also use $\mathbf{1}$ to denote the $N-1$ dimensional column vector whose entries are all ones. In matrix vector notation, the first $N-1$ components of the solution to Eq. (27) may be rewritten as

$$\begin{aligned} d\tilde{p}_t &= D(\tilde{p}_t) \left[\vartheta_{1:N-1}(t, H_t)' - \mathbf{1} \tilde{\vartheta}(t, \tilde{p}_t, H_t)' \right] (\Sigma_Y \Sigma_Y')^{-1} \Sigma_Y d\tilde{W}_t + \tilde{\beta}(t, \tilde{p}_t) dt \\ &\quad + \gamma D(\tilde{p}_t) \left(\vartheta_{1:N-1}(t, H_t)' - \mathbf{1} \tilde{\vartheta}(t, \tilde{p}_t, H_t)' \right) \pi_t dt + D(\tilde{p}_{t-}) \frac{1}{\tilde{h}(\tilde{p}_{t-})} \left(h^\perp - \mathbf{1} \tilde{h}(\tilde{p}_{t-}) \right) d\tilde{\xi}_t. \end{aligned}$$

Next, let us define

$$\begin{aligned} \alpha(t, \tilde{p}_t, H_t) &:= D(\tilde{p}_t) \left[\vartheta_{1:N-1}(t, H_t)' - \mathbf{1} \tilde{\vartheta}(t, \tilde{p}_t, H_t)' \right] (\Sigma_Y \Sigma_Y')^{-1} \Sigma_Y, \\ \beta_\gamma(t, \tilde{p}_t, H_t, \pi_t) &:= \tilde{\beta}(t, \tilde{p}_t) + \gamma \alpha(t, \tilde{p}_t, H_t) \Sigma_Y' \pi_t, \\ \varrho(\tilde{p}_{t-}) &:= D(\tilde{p}_{t-}) \frac{1}{\tilde{h}(\tilde{p}_{t-})} \left(h^\perp - \mathbf{1} \tilde{h}(\tilde{p}_{t-}) \right). \end{aligned}$$

Then, the dynamics of $\tilde{p}_s = (\tilde{p}_s^1, \dots, \tilde{p}_s^{N-1})'$ for $s \in [t, T]$, under the probability measure $\tilde{\mathbb{P}}$, is given by

$$d\tilde{p}_s = \beta_\gamma(s, \tilde{p}_s, H_s, \pi_s) ds + \alpha(s, \tilde{p}_s, H_s) d\tilde{W}_s + \varrho(\tilde{p}_{s-}) d\tilde{\xi}_s, \quad \tilde{p}_t = \tilde{p} \in \tilde{\Delta}_{N-1}. \quad (32)$$

Note that π affects the evolution of \tilde{p}_s through the drift β_γ , and also through the measure $\tilde{\mathbb{P}}$ due to an admissibility constraint analogous to (29).

Next, for a generic $0 \leq t \leq T$, such that $V_t = v$, $\tilde{p}_t = \tilde{p}$, and $H_t = z$ (recall that $z \in \{0, 1\}$ depending on whether or not default has occurred), we set

$$J(t, v, \tilde{p}, z, \pi; T) := \frac{v^\gamma}{\gamma} G(t, \tilde{p}, z, \pi) \quad \text{with} \quad G(t, \tilde{p}, z, \pi) := \mathbb{E}_{\tilde{\mathbb{P}}, \tilde{p}, z}^{\tilde{\mathbb{P}}} \left[e^{-\gamma \int_t^T \tilde{\eta}(s, \tilde{p}_s, H_s, \pi_s) ds} \right], \quad (33)$$

where, recalling the notation introduced in (30),

$$\tilde{\eta}(s, \tilde{p}_s, H_s, \pi_s) = \eta(s, e_N, H_s, \pi_s) + \sum_{i=1}^{N-1} (\eta(s, e_i, H_s, \pi_s) - \eta(s, e_N, H_s, \pi_s)) \tilde{p}_s^i.$$

Next, we define the value function

$$w(t, \tilde{p}, z) := \sup_{\pi \in \mathcal{A}(t, T; \tilde{p}, z)} \log(G(t, \tilde{p}, z, \pi)), \quad (34)$$

where the supremum above is over a suitable class $\mathcal{A}(t, T; \tilde{p}, z)$ of Markov (or feedback) admissible strategies $\pi_s := \pi(s, \tilde{p}_{s-}, H_{s-})$, that will be specified in detail later on (see Theorems 5.1 and 5.2 below for the details). Note that

$$\sup_{\pi \in \mathcal{A}(0, T; \tilde{p}, z)} J(v; \pi; T) = \frac{v^\gamma}{\gamma} \sup_{\pi} \mathbb{E}^{\tilde{\mathbb{P}}} \left[\hat{L}_T \right] = \frac{v^\gamma}{\gamma} e^{w(0, p_0^1, \dots, p_0^{N-1}, H_0)}, \quad (35)$$

where the last equality follows immediately using Eq. (28) and the fact that $\eta(s, p_s, H_s, \pi_s) = \tilde{\eta}(s, \tilde{p}_s, H_s, \pi_s)$.

We now proceed to derive the HJB equation corresponding to the value function in Eq. (34). Before doing so, we need to compute the generator \mathcal{L} of the Markov process $s \in [t, T] \rightarrow (s, \tilde{p}_s, H_s)$. This is done in the following lemma. Below and hereafter, we use $\mathbf{y} \cdot \mathbf{h}$ to denote componentwise multiplication of two vectors \mathbf{y} and \mathbf{h} .

Lemma 4.1. *Let $(\tilde{p}_s)_{s \in [t, T]}$ be the process in (32) with π of the form $\pi_s := \pi(s, \tilde{p}_{s-}, H_{s-})$, for a suitable function $\pi(s, \tilde{p}, z)$ such that (32) admits a unique solution. Then, for any $f(t, \tilde{p}, z)$ such that $f(t, \tilde{p}, 1)$ and $f(t, \tilde{p}, 0)$ are both $C^{1,2}$ -functions, we have*

$$f(s, \tilde{p}_s, H_s) = f(t, \tilde{p}_t, H_t) + \int_t^s \tilde{\mathcal{L}}f(u, \tilde{p}_u, H_u) du + \tilde{M}_s(f), \quad s \in (t, T], \quad (36)$$

where, denoting $\nabla_{\tilde{p}} f(t, \tilde{p}, z) := (\frac{\partial f}{\partial \tilde{p}^1}, \dots, \frac{\partial f}{\partial \tilde{p}^{N-1}})$, $f_t(t, \tilde{p}, z) := \frac{\partial f}{\partial t}$, and $D^2 f := \left[\frac{\partial^2 f}{\partial \tilde{p}^i \partial \tilde{p}^j} \right]_{i,j=1}^{N-1}$ and recalling the notation $\tilde{h}(\tilde{p}) := h^N + \sum_{i=1}^{N-1} (h_i - h_N) \tilde{p}^i$ and $h^\perp := (h_1, \dots, h_{N-1})'$,

$$\begin{aligned} \tilde{\mathcal{L}}f(t, \tilde{p}, z) &:= f_t(t, \tilde{p}, z) + \nabla_{\tilde{p}} f \beta_\gamma(t, \tilde{p}, z, \pi(t, \tilde{p}, z)) + \frac{1}{2} \text{tr}(\alpha \alpha' D^2 f) \\ &+ (1 - z) \left(f \left(t, \frac{1}{\tilde{h}(\tilde{p})} (\tilde{p} \cdot h^\perp), 1 \right) - f(t, \tilde{p}, 0) \right) \tilde{h}(\tilde{p}), \end{aligned}$$

and the $\tilde{\mathbb{P}}$ -local martingale component is

$$\tilde{M}_s(f) = \int_t^s \nabla_{\tilde{p}} f \alpha(u, \tilde{p}_u, H_u) d\tilde{W}_u + \int_s^t \left(f \left(u, \frac{1}{\tilde{h}(\tilde{p}_{u-})} (\tilde{p}_{u-} \cdot h^\perp), 1 \right) - f(u, \tilde{p}_{u-}, 0) \right) d\tilde{\xi}_u. \quad (37)$$

Proof. Let $\tilde{p}^{c,i}$ denote the continuous component of \tilde{p}^i , determined by the first two terms on the right-hand side of Eq. (32). Using Itô's formula, we have

$$\begin{aligned} f(s, \tilde{p}_s, H_s) &= f(u, \tilde{p}_t, H_t) + \int_t^s f_t(u, \tilde{p}_u, H_u) du + \sum_{i=1}^{N-1} \int_t^s \frac{\partial f}{\partial \tilde{p}^i} d\tilde{p}_u^{c,i} + \frac{1}{2} \sum_{i,j=1}^{N-1} \int_t^s \frac{\partial^2 f}{\partial \tilde{p}^i \partial \tilde{p}^j} d\langle \tilde{p}^{c,i}, \tilde{p}^{c,j} \rangle_u \\ &+ \sum_{t < u \leq s} (f(u, \tilde{p}_u, H_u) - f(u, \tilde{p}_{u-}, H_{u-})). \end{aligned} \quad (38)$$

Note that the size of the jump of p_t^i at the default time τ is given by

$$\tilde{p}_\tau^i - \tilde{p}_{\tau-}^i = \tilde{p}_{\tau-}^i \frac{h_i - \tilde{h}(\tilde{p}_{\tau-})}{\tilde{h}(\tilde{p}_{\tau-})}, \quad (39)$$

thus implying that $\tilde{p}_\tau^i = \tilde{p}_{\tau-}^i h_i / \tilde{h}(\tilde{p}_{\tau-})$ and $\tilde{p}_\tau = (1/\tilde{h}(\tilde{p}_{\tau-}))(\tilde{p}_{\tau-} \cdot h^\perp)$. For $t < \tau \leq s$, this leads to

$$\begin{aligned} \sum_{t < u \leq s} (f(u, \tilde{p}_u, H_u) - f(u, \tilde{p}_{u-}, H_{u-})) &= \left[f\left(\tau, \frac{1}{\tilde{h}(\tilde{p}_{\tau-})}(\tilde{p}_{\tau-} \cdot h^\perp), 1\right) - f(\tau, \tilde{p}_{\tau-}, 0) \right] (H_s - H_t) \\ &= \int_t^s \left(f\left(u, \frac{1}{\tilde{h}(\tilde{p}_{u-})}(\tilde{p}_{u-} \cdot h^\perp), 1\right) - f(u, \tilde{p}_{u-}, 0) \right) dH_u \\ &= \int_t^s \left(f\left(u, \frac{1}{\tilde{h}(\tilde{p}_{u-})}(\tilde{p}_{u-} \cdot h^\perp), 1\right) - f(u, \tilde{p}_{u-}, 0) \right) (d\tilde{\xi}_u + \bar{H}_u - \tilde{h}(\tilde{p}_{u-}) du), \end{aligned}$$

where in the last equality we had used Eq. (26) and the fact that $\hat{h}_u = \sum_{i=1}^N h_i p_u^i = h_N + \sum_{i=1}^{N-1} (h_i - h_N) p_u^i = \tilde{h}(\tilde{p}_u)$. From this, we deduce that Eq. (38) may be rewritten as

$$\begin{aligned} f(s, \tilde{p}_s, H_s) &= f(t, \tilde{p}_t, H_t) + \int_t^s f_u(u, \tilde{p}_u, H_u) du + \int_t^s \nabla_{\tilde{p}} f \beta_\gamma(u, \tilde{p}_u, H_u, \pi_u) du + \frac{1}{2} \sum_{i,j=1}^{N-1} \int_t^s (\alpha \alpha')_{ij} \frac{\partial^2 f}{\partial \tilde{p}^i \partial \tilde{p}^j} du \\ &\quad + \int_t^s \nabla_{\tilde{p}} f \alpha(u, \tilde{p}_u, H_u) d\tilde{W}_u + \int_t^s \left(f\left(u, \frac{1}{\tilde{h}(\tilde{p}_{u-})}(\tilde{p}_{u-} \cdot h^\perp), 1\right) - f(u, \tilde{p}_{u-}, 0) \right) d\tilde{\xi}_u \\ &\quad + \int_t^s \left(f\left(u, \frac{1}{\tilde{h}(\tilde{p}_u)}(\tilde{p}_u \cdot h^\perp), 1\right) - f(u, \tilde{p}_u, 0) \right) (1 - H_u) \tilde{h}(\tilde{p}_u) du, \end{aligned} \quad (40)$$

which proves the lemma. \square

We are now ready to derive the HJB equation associated to the control problem based on standard heuristic arguments. First, in light of the dynamic programming principle, we expect that, for any $s \in (t, T]$,

$$w(t, \tilde{p}, z) = \sup_{\pi \in \mathcal{A}(t, T)} \log \mathbb{E}_{t, \tilde{p}, z}^{\tilde{\mathbb{P}}} \left[e^{w(s, \tilde{p}_s, H_s) - \gamma \int_t^s \tilde{\eta}(u, \tilde{p}_u, H_u, \pi_u) du} \right]. \quad (41)$$

Next, define $\varepsilon(s, \tilde{p}, z) = e^{w(s, \tilde{p}, z)}$ and note that, in light of Lemma 4.1,

$$\varepsilon(s, \tilde{p}_s, H_s) = \varepsilon(t, \tilde{p}_t, H_t) + \int_t^s \tilde{\mathcal{L}}\varepsilon(u, \tilde{p}_u, H_u) du + \tilde{M}_s(\varepsilon),$$

where the last term $\tilde{M}_s(\varepsilon)$ represents the local martingale component of $\varepsilon(s, \tilde{p}_s, H_s)$. Plugging the previous equation into (41), we expect the following relation to hold:

$$0 = \sup_{\pi} \mathbb{E}_{t, \tilde{p}, z}^{\tilde{\mathbb{P}}} \left[\varepsilon(t, \tilde{p}_t, H_t) \left(e^{-\gamma \int_t^s \tilde{\eta}(u, \tilde{p}_u, H_u, \pi_u) du} - 1 \right) + e^{-\gamma \int_t^s \tilde{\eta}(u, \tilde{p}_u, H_u, \pi_u) du} \int_t^s \tilde{\mathcal{L}}\varepsilon(u, \tilde{p}_u, H_u) du \right],$$

assuming that the local martingale component is a true martingale. Dividing by $s - t$ and taking the limit of the above expression as $s \rightarrow t$ leads us to the HJB equation:

$$0 = \sup_{\pi} \left[(\tilde{\mathcal{L}} - \gamma \tilde{\eta}(t, \tilde{p}, z, \pi)) \varepsilon(t, \tilde{p}, z) \right]. \quad (42)$$

Let us write (42) in terms of w . To this end, let us denote the differential component of $\tilde{\mathcal{L}}$ as $\tilde{\mathcal{D}}$; i.e.,

$$\tilde{\mathcal{D}}f(t, \tilde{p}, z) := f_t(t, \tilde{p}, z) + \nabla_{\tilde{p}} f(t, \tilde{p}, z) \beta_\gamma(t, \tilde{p}, z, \pi(t, \tilde{p}, z)) + \frac{1}{2} \text{tr}(\alpha \alpha' D^2 f).$$

Then, we note that

$$\begin{aligned} \tilde{\mathcal{L}}\varepsilon(t, \tilde{p}, z) &= \tilde{\mathcal{D}}\varepsilon(t, \tilde{p}, z) + (1 - z) \tilde{h}(\tilde{p}) \left(e^{w\left(t, \frac{1}{\tilde{h}(\tilde{p})} \tilde{p} \cdot h^\perp, 1\right)} - e^{w(t, \tilde{p}, 0)} \right) \\ &= e^{w(t, \tilde{p}, z)} \left(\tilde{\mathcal{D}}w + \frac{1}{2} \|\nabla_{\tilde{p}} w \alpha\|^2 + (1 - z) \tilde{h}(\tilde{p}) \left[e^{w\left(t, \frac{1}{\tilde{h}(\tilde{p})} \tilde{p} \cdot h^\perp, 1\right) - w(t, \tilde{p}, 0)} - 1 \right] \right). \end{aligned} \quad (43)$$

Thus, Eq. (42) takes the form:

$$0 = \sup_{\pi} \left[\varepsilon(t, \tilde{p}, z) \left(\tilde{D}w + \frac{1}{2} \|\nabla_{\tilde{p}} w \alpha\|^2 + (1-z)\tilde{h}(\tilde{p}) \left[e^{w(t, \frac{1}{h(\tilde{p})} \tilde{p} \cdot h^\perp, 1) - w(t, \tilde{p}, 0)} - 1 \right] - \gamma \tilde{\eta}(t, \tilde{p}, z, \pi) \right) \right]. \quad (44)$$

In order to get a more explicit form, let us recall that

$$\eta(t, e_i, z, \pi) = -r + \pi^S(r - \langle \mu, e_i \rangle) + \pi^P \left(r - \frac{1}{2} v^2 - a(t, e_i, z) \right) + \frac{1-\gamma}{2} \pi' \Sigma_Y' \Sigma_Y \pi,$$

and note that

$$\begin{aligned} \tilde{\eta}(t, \tilde{p}, z, \pi) &= \eta(t, e_N, z, \pi(t, \tilde{p}, z)) + \sum_{i=1}^{N-1} (\eta(t, e_i, z, \pi(t, \tilde{p}, z)) - \eta(t, e_N, z, \pi(t, \tilde{p}, z))) \tilde{p}^i \\ &= -r + \pi^S(r - \tilde{\mu}(\tilde{p})) + \pi^P \left(r - \frac{1}{2} v^2 - \tilde{a}(t, \tilde{p}, z) \right) + \frac{1-\gamma}{2} \pi' \Sigma_Y' \Sigma_Y \pi. \end{aligned} \quad (45)$$

We can now rewrite Eq. (44) as

$$\begin{aligned} \frac{\partial w}{\partial t} + \frac{1}{2} \text{tr}(\alpha \alpha' D^2 w) + \frac{1}{2} (\nabla_{\tilde{p}} w) \alpha \alpha' (\nabla_{\tilde{p}} w)' + \gamma r + (1-z)\tilde{h}(\tilde{p}) \left[e^{w(t, \frac{1}{h(\tilde{p})} \tilde{p} \cdot h^\perp, 1) - w(t, \tilde{p}, 0)} - 1 \right] \\ + \sup_{\pi} \left\{ (\nabla_{\tilde{p}} w) \beta_{\gamma} - \gamma \pi^S(r - \tilde{\mu}(\tilde{p})) - (1-z)\gamma \pi^P \left(r - \frac{1}{2} v^2 - \tilde{a}(t, \tilde{p}, z) \right) - \frac{1}{2} \gamma (1-\gamma) \pi' \Sigma_Y' \Sigma_Y \pi \right\} = 0, \end{aligned} \quad (46)$$

with terminal condition $w(T, \tilde{p}, z) = 0$.

Depending on whether or not default has occurred, we will have two separate optimization problems to solve. Indeed, after default has occurred, the investor cannot invest in the defaultable bond, and only allocates his wealth in the stock and risk-free asset. The next section analyzes in detail the two cases.

5 Solution to the Optimal Control Problem

We analyze the control problem developed in the previous section. We first decompose it into two related optimization subproblems: the post and the pre-default problems. As we will demonstrate, in order to solve the pre-default optimization subproblem, we need the solution of the post-default one.

5.1 Post Default Optimization Problem

Assume that default has already occurred; i.e., we are at a time t so that $\tau < t$. In particular, this means that $\pi_t^P = 0$. Let us denote by $\underline{w}(t, \tilde{p}) := w(t, \tilde{p}, 1)$ the value function in the post-default optimization problem. Then, we may rewrite Eq. (46) as follows:

$$\begin{aligned} 0 &= \underline{w}_t + \frac{1}{2} \text{tr}(\underline{\alpha} \underline{\alpha}' D^2 \underline{w}) + \frac{1}{2} (\nabla_{\tilde{p}} \underline{w}) \underline{\alpha} \underline{\alpha}' (\nabla_{\tilde{p}} \underline{w})' + \gamma r \\ &\quad + \sup_{\pi^S} \left[(\nabla_{\tilde{p}} \underline{w}) \underline{\beta}_{\gamma} - \gamma \pi^S(r - \tilde{\mu}(\tilde{p})) - \frac{\sigma^2}{2} \gamma (1-\gamma) (\pi^S)^2 \right], \end{aligned} \quad (47)$$

$$\underline{w}(T, \tilde{p}) = 0. \quad (48)$$

Here, $\underline{\alpha}(\tilde{p})$ is a $(N-1) \times 1$ vector determined by the first column of $\alpha(t, \tilde{p}, 1)$ (the second column of $\alpha(t, \tilde{p}, 1)$ consists of all zeros). Concretely,

$$\underline{\alpha}(\tilde{p}) := [D(\tilde{p}) (\underline{\vartheta}' - \mathbf{1} \vartheta(\tilde{p}))] \frac{1}{\sigma}, \quad (49)$$

where $\underline{\vartheta} = (\mu_1 - \frac{1}{2}\sigma^2, \dots, \mu_{N-1} - \frac{1}{2}\sigma^2)$ is the first row of $\vartheta_{1:N-1}(t, 1)$ (the second row consists of all zeros) and, correspondingly, $\underline{\vartheta}(\tilde{p}) = \tilde{\mu}(\tilde{p}) - \frac{1}{2}\sigma^2$ is a scalar with $\tilde{\mu}(\tilde{p}) = \mu_N + \sum_{i=1}^{N-1}(\mu_i - \mu_N)\tilde{p}^i$. Similarly, $\underline{\beta}_\gamma(t, \tilde{p}, \pi)$ in (47) is defined as

$$\underline{\beta}_\gamma(t, \tilde{p}, \pi) := \beta_\gamma(t, \tilde{p}, 1, \pi) = \tilde{\beta}(t, \tilde{p}) + \gamma\sigma\pi^S \underline{\alpha}(\tilde{p}),$$

where we recall that

$$\tilde{\beta}(t, \tilde{p}) = \left(\varpi_{N,1}(t) + \sum_{\ell=1}^{N-1} [\varpi_{\ell,1}(t) - \varpi_{N,1}(t)]\tilde{p}^\ell, \dots, \varpi_{N,N-1}(t) + \sum_{\ell=1}^{N-1} [\varpi_{\ell,N-1}(t) - \varpi_{N,N-1}(t)]\tilde{p}^\ell \right)'.$$

It can easily be checked that the minimizer of Eq. (47) is given by

$$\pi^S = \frac{1}{\sigma^2(1-\gamma)} \{ \tilde{\mu}(\tilde{p}) - r + \sigma(\nabla_{\tilde{p}} \underline{w}) \underline{\alpha} \}. \quad (50)$$

Plugging the maximizer (50) in (47), we obtain

$$\underline{w}_t + \frac{1}{2} \text{tr}(\underline{\alpha} \underline{\alpha}' D^2 \underline{w}) + \frac{1}{2(1-\gamma)} (\nabla_{\tilde{p}} \underline{w}) \underline{\alpha} \underline{\alpha}' (\nabla_{\tilde{p}} \underline{w})' + (\nabla_{\tilde{p}} \underline{w}) \underline{\Phi} + \underline{\Psi} = 0, \quad (51)$$

$$\underline{w}(T, \tilde{p}) = 0, \quad (52)$$

where

$$\begin{aligned} \underline{\Phi}(t, \tilde{p}) &= \tilde{\beta}(t, \tilde{p}) + \frac{\gamma}{1-\gamma} \frac{\tilde{\mu}(\tilde{p}) - r}{\sigma} \underline{\alpha}(\tilde{p}), \\ \underline{\Psi}(t, \tilde{p}) &= \gamma r + \frac{\gamma}{2(1-\gamma)} \left(\frac{\tilde{\mu}(\tilde{p}) - r}{\sigma} \right)^2. \end{aligned}$$

The following result develops a detailed analysis of the post-default optimization problem. Below, we denote the interior of a set $A \in \mathbb{R}^{N-1}$ by A° and we set

$$\mathcal{C}^{1,2} := C^{1,2} \left((0, T) \times \tilde{\Delta}_{N-1}^\circ \right) \cap C \left([0, T] \times \tilde{\Delta}_{N-1} \right).$$

Theorem 5.1. *The following assertions hold true:*

- (1) *For any $T \in (0, T)$, there exists a function $\underline{w} \in \mathcal{C}^{1,2}$ that solves the Dirichlet problem (51)-(52).*
- (2) *The solution $\underline{w}(t, \tilde{p})$ of (51)-(52) coincides with the optimal value function $w(t, \tilde{p}, 1)$ introduced in (34), when $\mathcal{A}_t(t, T; \tilde{p}, 1)$ is constrained to the class of feedback controls $\pi_s^S = \pi^S(s, \tilde{p}_s)$ and $\pi_s^P \equiv 0$ such that the SDE (32) admits a unique solution.*
- (3) *The optimal feedback control $\{\pi_s^S\}_{s \in [t, T]}$, denoted by $\tilde{\pi}_s^S$, can be written as $\tilde{\pi}_s^S = \tilde{\pi}^S(s, \tilde{p}_s)$ with*

$$\tilde{\pi}^S(s, \tilde{p}) := \frac{1}{\sigma^2(1-\gamma)} (\tilde{\mu}(\tilde{p}) - r + \sigma \nabla_{\tilde{p}} \underline{w}(s, \tilde{p}) \underline{\alpha}). \quad (53)$$

The proof of Theorem 5.1 is reported in Appendix B. For now, let us mention a few useful remarks about the proof of point (1). The proof of this result follows from the Feynman-Kac formula as outlined in, e.g., the proof of Theorem 3.1 in Tamura and Watanabe (2011) (see also Nagai and Runggaldier (2008)). Indeed, the idea therein is to transform the problem into a linear PDE via the Hopf-Cole transformation:

$$\underline{\psi}(t, \tilde{p}) = e^{\frac{1}{1-\gamma} \underline{w}(t, \tilde{p})}. \quad (54)$$

Then, it follows that $\underline{w}(t, \tilde{p})$ solves Eq. (51)-(52) if and only if $\underline{\psi}(t, \tilde{p})$ solves the linear PDE

$$\begin{aligned} \frac{\partial \underline{\psi}}{\partial t} + \frac{1}{2} \text{tr}(\underline{\alpha} \underline{\alpha}' D^2 \underline{\psi}) + \underline{\Phi}' \nabla_{\tilde{p}} \underline{\psi} + \frac{\underline{\Psi}}{1-\gamma} \underline{\psi} &= 0, \\ \underline{\psi}(T, \tilde{p}) &= 1. \end{aligned} \quad (55)$$

It is illustrative to setup explicitly the problem in the case of two hidden regimes. In that case, the vector $\tilde{p} := p$ becomes one dimensional, with p denoting the filter probability that the Markov chain is in regime “1”, and $\underline{\psi}(t, p)$ is one dimensional with $t \in \mathbb{R}^+$ and $0 \leq p \leq 1$. We then have that Eq. (55) reduces to

$$\begin{aligned} \frac{\partial \underline{\psi}}{\partial t} + \frac{1}{2} \underline{\alpha}^2 \frac{\partial^2 \underline{\psi}(t, p)}{\partial p^2} + \underline{\Phi}(t, p) \frac{\partial \underline{\psi}(t, p)}{\partial p} + \underline{\Psi}(t, p) \frac{\underline{\psi}(t, p)}{1-\gamma} &= 0, \\ \underline{\psi}(T, p) &= 1, \end{aligned}$$

where

$$\begin{aligned} \underline{\alpha}(t, p) &= \sigma^{-1} p(1-p)(\mu_1 - \mu_2), \\ \beta(t, p) &= \varpi_{11} p + \varpi_{21}(1-p), \\ \underline{\Phi}(t, p) &= \beta(t, p) + \frac{\gamma}{1-\gamma} \frac{\mu_1 p + (1-p)\mu_2 - r}{\sigma} \underline{\alpha}(t, p), \\ \underline{\Psi}(t, p) &= \gamma r + \frac{\gamma}{2(1-\gamma)} \left(\frac{\mu_1 p + (1-p)\mu_2 - r}{\sigma} \right)^2. \end{aligned}$$

5.2 Pre Default Optimization Problem

Assume that $\tau > t$, i.e. default has not occurred by time t . Let us denote by $\bar{w}(t, \tilde{p}) := w(t, \tilde{p}, 0)$ the value function in the pre-default optimization problem. Then, we may rewrite Eq. (46) as

$$\begin{aligned} \bar{w}_t + \frac{1}{2} \text{tr}(\bar{\alpha} \bar{\alpha}' D^2 \bar{w}) + \frac{1}{2} (\nabla_{\tilde{p}} \bar{w}) \bar{\alpha} \bar{\alpha}' (\nabla_{\tilde{p}} \bar{w})' + \\ \sup_{\pi=(\pi^S, \pi^P)'} \left\{ (\nabla_{\tilde{p}} \bar{w}) \bar{\beta}_\gamma - \gamma \pi^S (r - \tilde{\mu}(\tilde{p})) - \gamma \pi^P \left(r - \frac{1}{2} v^2 - \tilde{a}(t, \tilde{p}, 0) \right) - \frac{1}{2} \gamma (1-\gamma) \pi' \Sigma_Y' \Sigma_Y \pi \right\} \\ + \tilde{h}(\tilde{p}) \left[e^{\underline{w}(t, \frac{1}{h(\tilde{p})} \tilde{p} \cdot h^\perp) - \bar{w}(t, \tilde{p})} - 1 \right] + \gamma r = 0, \\ \bar{w}(T, \tilde{p}) = 0. \end{aligned} \quad (56)$$

Above, $\bar{\alpha}(t, p)$ is a $(N-1) \times 2$ matrix given by

$$\bar{\alpha}(t, \tilde{p}) := \alpha(t, \tilde{p}, 0) = [D(\tilde{p}) (\bar{\vartheta}(t)' - \mathbf{1} \bar{\vartheta}(t, \tilde{p})')] (\Sigma_Y \Sigma_Y')^{-1} \Sigma_Y,$$

with $\bar{\vartheta}(t)$ defined as the $2 \times (N-1)$ matrix $\vartheta_{1:N-1}(t, 0)$ and $\bar{\vartheta}(t, \tilde{p})$ defined by the two dimensional column vector $\tilde{\vartheta}(t, \tilde{p}, 0)$. Further,

$$\bar{\beta}_\gamma(t, \tilde{p}, \pi) := \beta_\gamma(t, \tilde{p}, \pi, 0) = \tilde{\beta}(t, \tilde{p}) + \gamma \bar{\alpha}(t, \tilde{p}) \Sigma_Y' \pi.$$

It is important to point out the explicit appearance of the post-default value function \underline{w} in the PDE (56) satisfied by the pre-default value function \bar{w} . This establishes the required relationship between pre and post-default optimization subproblems.

Next, define

$$\Upsilon(t, \tilde{p}) = \left(r - \tilde{\mu}(\tilde{p}), r - \frac{1}{2} v^2 - \tilde{a}(t, \tilde{p}, 0) \right)'.$$

Then, we can rewrite Eq. (56) as

$$\begin{aligned} & \bar{w}_t + \frac{1}{2} \text{tr}(\bar{\alpha} \bar{\alpha}' D^2 \bar{w}) + \frac{1}{2} (\nabla_{\tilde{p}} \bar{w}) \bar{\alpha} \bar{\alpha}' (\nabla_{\tilde{p}} \bar{w})' + \\ & \sup_{\pi} \left\{ (\nabla_{\tilde{p}} \bar{w}) \bar{\beta}_{\gamma} - \gamma \pi' \Upsilon - \frac{1}{2} \gamma (1 - \gamma) \pi' \Sigma_Y' \Sigma_Y \pi \right\} + \tilde{h}(\tilde{p}) \left[e^{\underline{w}(t, \frac{1}{h(\tilde{p})} \tilde{p} \cdot h^{\perp}) - \bar{w}(t, \tilde{p})} - 1 \right] + \gamma r = 0 \\ & \bar{w}(T, \tilde{p}) = 0. \end{aligned} \quad (57)$$

Differentiating the above expression with respect to π , we obtain that the maximal point π^* is the solution of the following equation:

$$\gamma \Sigma_Y \bar{\alpha}' (\nabla_{\tilde{p}} \bar{w})' - \gamma \Upsilon - \gamma (1 - \gamma) \Sigma_Y' \Sigma_Y \pi^* = 0.$$

Solving the previous equation for π^* yields:

$$\pi^* = \frac{1}{1 - \gamma} (\Sigma_Y' \Sigma_Y)^{-1} (-\Upsilon + \Sigma_Y \bar{\alpha}' (\nabla_{\tilde{p}} \bar{w})'). \quad (58)$$

After plugging π^* into (57), and performing algebraic simplifications (see Appendix B for details), we obtain

$$\bar{w}_t + \frac{1}{2} \text{tr}(\bar{\alpha} \bar{\alpha}' D^2 \bar{w}) + \frac{1}{2(1 - \gamma)} (\nabla_{\tilde{p}} \bar{w}) \bar{\alpha} \bar{\alpha}' (\nabla_{\tilde{p}} \bar{w})' + (\nabla_{\tilde{p}} \bar{w}) \bar{\Phi} + \tilde{h}(\tilde{p}) e^{\underline{w}(t, \frac{1}{h(\tilde{p})} \tilde{p} \cdot h^{\perp})} e^{-\bar{w}(t, \tilde{p})} + \bar{\Psi} = 0, \quad (59)$$

$$\bar{w}(T, \tilde{p}) = 0, \quad (60)$$

where

$$\begin{aligned} \bar{\Phi}(t, \tilde{p}) &= \tilde{\beta}(t, \tilde{p}) - \frac{\gamma}{1 - \gamma} \bar{\alpha} \Sigma_Y^{-1} \Upsilon, \\ \bar{\Psi}(t, \tilde{p}) &= \frac{1}{2} \frac{\gamma}{1 - \gamma} \Upsilon' (\Sigma_Y' \Sigma_Y)^{-1} \Upsilon + \gamma r - \tilde{h}(\tilde{p}). \end{aligned}$$

The following result shows a verification theorem for the pre default optimization problem.

Theorem 5.2. *Suppose that the conditions of Theorem 5.1 are satisfied and, in particular, we let $\underline{w} \in \mathcal{C}^{1,2}$ be the solution of (51) with boundary condition (52). Additionally, assume that there exists a function $\bar{w} \in \mathcal{C}^{1,2}$ solving the Dirichlet problem (59)-(60) Then, the following assertions hold true:*

- (1) *The solution $\bar{w}(t, \tilde{p})$ coincides with the optimal value function $w(t, \tilde{p}, 0)$ introduced in (34), when $\mathcal{A}_t(t, T; \tilde{p}, 0)$ is constrained to the class of feedback controls $\pi_s^S = \pi^S(s, \tilde{p}_{s-}, H_{s-})$ and $\pi_s^P = \pi^P(s, \tilde{p}_{s-}, H_{s-})$ such that $\pi^P(s, \tilde{p}, 1) \equiv 0$ and the SDE (32) admits a unique solution.*
- (2) *The optimal feedback controls $\{\pi_s\}_{s \in [t, T]} := \{(\pi_s^S, \pi_s^P)'\}_{s \in [t, T]}$, denoted by $\tilde{\pi} := (\tilde{\pi}^S, \tilde{\pi}^P)'$, can be written as $\tilde{\pi}_s^S = \tilde{\pi}^S(s, \tilde{p}_{s-}, H_{s-})$ and $\tilde{\pi}_s^P = \tilde{\pi}^P(s, \tilde{p}_{s-}, H_{s-})$ with*

$$(\tilde{\pi}^S(s, \tilde{p}, 0), \tilde{\pi}^P(s, \tilde{p}, 0))' := \frac{1}{1 - \gamma} (\Sigma_Y' \Sigma_Y)^{-1} (-\Upsilon + \Sigma_Y \bar{\alpha}' \nabla_{\tilde{p}} \bar{w}(s, \tilde{p})), \quad (61)$$

$$(\tilde{\pi}^S(s, \tilde{p}, 1), \tilde{\pi}^P(s, \tilde{p}, 1))' := \left(\frac{1}{\sigma^2(1 - \gamma)} (\tilde{\mu}(\tilde{p}) - r + \sigma \bar{\alpha}' \nabla_{\tilde{p}} \underline{w}(s, \tilde{p}')), 0 \right)'. \quad (62)$$

The proof of Theorem 5.2 is reported in Appendix B. Proving that the Dirichlet problem (59) admits a classical solution is not as direct as in the post-default case. Still, the Hopf-Cole transformation reveals the type of PDE that is encountered when solving for the optimal value function. Concretely, it follows that the function $\bar{w}(t, \tilde{p})$ solves the problem (59) if and only if the function

$$\bar{\psi}(t, \tilde{p}) = e^{\frac{1}{1 - \gamma} \bar{w}(t, \tilde{p})}, \quad (63)$$

solves the Dirichlet problem

$$\bar{\psi}_t + \frac{1}{2} \text{tr}(\bar{\alpha} \bar{\alpha}' D^2 \bar{\psi}) + (\nabla_{\bar{p}} \bar{\psi}) \bar{\Phi} + \bar{\Psi} \frac{\bar{\psi}(t, \bar{p})}{1 - \gamma} + \tilde{h}(\bar{p}) e^{w(t, \frac{1}{h(\bar{p})} \bar{p} \cdot h^\perp)} \frac{\bar{\psi}(t, \bar{p})^\gamma}{1 - \gamma} = 0, \quad \bar{\psi}(T, \bar{p}) = 1. \quad (64)$$

Eq. (64) shows that optimal pre-default value function satisfies a degenerate nonlinear parabolic PDE. The degeneracy comes from the fact that $\bar{\alpha} \bar{\alpha}'$ vanishes on the boundary of $\tilde{\Delta}_{N-1}$, hence the differential operator is not uniformly elliptic on the boundary. A detailed analysis of the global existence of a solution is a nontrivial matter falling outside the scope of this paper. We refer the reader to Goldstein et al. (2003) for related discussions.

For illustration purposes, let us write explicitly the previous problem for the case of two hidden regimes. In that case, setting for simplicity $p = \tilde{p} = p_1$, (64) takes the form:

$$\begin{aligned} \frac{\partial \bar{\psi}}{\partial t} + \frac{1}{2} \bar{\alpha} \bar{\alpha}' \frac{\partial^2 \bar{\psi}}{\partial p^2} + \bar{\Phi}(t, p) \frac{\partial \bar{\psi}}{\partial p} + \bar{\Psi}(t, p) \frac{\bar{\psi}(t, p)}{1 - \gamma} + (h_2 + (h_1 - h_2)p) e^{w(t, \frac{ph_1}{h_2 + (h_1 - h_2)p})} \frac{\bar{\psi}(t, p)^\gamma}{1 - \gamma} &= 0, \\ \bar{\psi}(T, p) &= 1, \end{aligned}$$

where, in terms of $\bar{a}(t, e_i) := a(t, e_i, 0)$,

$$\begin{aligned} \bar{\alpha}(t, p) &= p(1 - p) \left[\frac{\mu_1 - \mu_2}{\sigma}, \frac{\bar{a}(t, e_1) - \bar{a}(t, e_2)}{v} \right] := [\bar{\alpha}_{11}(t, p), \bar{\alpha}_{12}(t, p)] \\ \bar{\Phi}(t, p) &= \varpi_{21} + (\varpi_{11} - \varpi_{21})p + \frac{\gamma}{1 - \gamma} \left(\frac{\bar{\alpha}_{11}(t, p)}{\sigma} (\tilde{\mu}(p) - r) + \frac{\bar{\alpha}_{12}(t, p)}{v} \left(\frac{1}{2} v^2 - r + \tilde{a}(t, p) \right) \right) \\ \bar{\Psi}(t, p) &= \frac{1}{2} \frac{\gamma}{1 - \gamma} \left(\frac{(\mu_2 - r + (\mu_1 - \mu_2)p)^2}{\sigma^2} + \frac{(\frac{1}{2} v^2 - r + \bar{a}(t, e_2) + (\bar{a}(t, e_1) - \bar{a}(t, e_2))p)^2}{v^2} \right) + \gamma r - h_2 + (h_1 - h_2)p. \end{aligned}$$

6 Conclusions

We studied the optimal investment problem of a power investor in an economy consisting of a defaultable bond, a stock, and a money market account. The price processes of these securities are assumed to have drift coefficients modulated by a hidden Markov chain. We have reduced the partially observed stochastic control problem to a risk sensitive one, where the state is given by the filtered regime probabilities, derived using the reference probability approach to filtering. The conditioning filtration, determined by the stock prices, the noisy bond price, and the indicator of default occurrence, is generated by both a Brownian component and a pure jump martingale. The filter has been used to derive the HJB partial differential equation corresponding to the risk sensitive control problem. Our methodology consisted of splitting the latter into a pre-default and a post-default dynamic programming subproblems. In line with the dynamic programming principle, the value function associated to the pre-default problem depends on the value function associated to the post-default counterpart. The PDE associated to the post-default value function can be transformed to a linear parabolic PDE, degenerating at the boundary, for which existence and uniqueness of a classical solution can be guaranteed. We have proven a verification theorem for the pre-default dynamic optimization subproblem, and illustrated that the PDE satisfied by the pre-default value function is of nonlinear degenerate parabolic type.

A Proofs related to Section 3

Proof of Lemma 3.1.

Let us introduce the following notation

$$H_t^i := \mathbf{1}_{\{X_t = e_i\}}.$$

Note that $X_t = (H_t^1, \dots, H_t^N)'$ and, from (1),

$$H_t^i = H_0^i + \int_0^t \sum_{\ell=1}^N \varpi_{\ell,i}(s) H_s^\ell ds + M_i(t). \quad (65)$$

From Eq. (18) and (13), we deduce that, under $\hat{\mathbb{P}}$,

$$dL_t = L_{t-}(h_t - 1)d\hat{\xi}_t + L_t Q'(t, X_t, H_t, \pi_t) dY_t - L_t \gamma \eta(t, X_t, H_t, \pi_t) dt$$

which yields that

$$[L, H^i]_t = \int_0^t L_{s-} Q'(s, X_s, H_s, \pi_s) d[Y, H^i]_s + \int_0^t L_{s-}(h_s - 1) d[\hat{\xi}, H^i]_s.$$

As $(Y_t)_{t \geq 0}$ and $(H_t)_{t \geq 0}$ are independent of $(X_t)_{t \geq 0}$ (and, hence, of H^i), under $\hat{\mathbb{P}}$, it holds (see also Wong and Hajek (1985)), $\hat{\mathbb{P}}$ almost surely, that

$$[Y, H^i]_s = [\hat{\xi}, H^i]_s = 0, \text{ for all } s \geq 0.$$

Thus, applying Itô's formula, we obtain

$$\begin{aligned} L_t H_t^i &= H_0^i + \int_0^t H_{s-}^i dL_s + \int_0^t L_{s-} dH_s^i \\ &= H_0^i + \int_0^t H_s^i L_s Q'(s, X_s, H_s, \pi_s) dY_s + \int_0^t H_{s-}^i L_{s-}(h_{s-} - 1) d\hat{\xi}_s \\ &\quad - \int_0^t H_s^i L_s \gamma \eta(s, X_s, H_s, \pi_s) ds + \int_0^t L_s \sum_{\ell=1}^N \varpi_{\ell,i}(s) H_s^\ell ds + \int_0^t L_{s-} dM_i(s) \end{aligned} \quad (66)$$

Since $(M_i(t))_{t \geq 0}$ is a $((\mathcal{F}_t^X)_{t \geq 0}, \hat{\mathbb{P}})$ -martingale, and \mathcal{G}_t^I is independent of \mathcal{F}_t^X under $\hat{\mathbb{P}}$, we have that $\mathbb{E}^{\hat{\mathbb{P}}} \left[\int_0^t L_{s-} dM_i(s) | \mathcal{G}_t^I \right] = 0$. Therefore, taking \mathcal{G}_t^I conditional expectations in Eq. (66), we obtain

$$\begin{aligned} \mathbb{E}^{\hat{\mathbb{P}}} [L_t H_t^i | \mathcal{G}_t^I] &= 1 + \int_0^t \mathbb{E}^{\hat{\mathbb{P}}} [L_s H_s^i Q'(s, e_i, H_s, \pi_s) | \mathcal{G}_s^I] dY_s + \int_0^t \mathbb{E}^{\hat{\mathbb{P}}} [L_{s-} H_{s-}^i (h_{s-} - 1) | \mathcal{G}_s^I] d\hat{\xi}_s \\ &\quad - \int_0^t \mathbb{E}^{\hat{\mathbb{P}}} [H_s^i L_s \gamma \eta(s, e_i, H_s, \pi_s) | \mathcal{G}_s^I] ds + \int_0^t \mathbb{E}^{\hat{\mathbb{P}}} \left[\sum_{\ell=1}^N \varpi_{\ell,i}(s) L_s H_s^\ell | \mathcal{G}_s^I \right] ds, \end{aligned} \quad (67)$$

where we have used that, if ϕ_t is \mathbb{G} predictable then, (see, for instance, Wong and Hajek (1985), Lemma 3.2),

$$\begin{aligned} \mathbb{E}^{\hat{\mathbb{P}}} \left[\int_0^t \phi_s L_{s-} dY_s | \mathcal{G}_t^I \right] &= \int_0^t \mathbb{E}^{\hat{\mathbb{P}}} [\phi_s L_{s-} | \mathcal{G}_s^I] dY_s \\ \mathbb{E}^{\hat{\mathbb{P}}} \left[\int_0^t \phi_s L_{s-} d\hat{\xi}_s | \mathcal{G}_t^I \right] &= \int_0^t \mathbb{E}^{\hat{\mathbb{P}}} [\phi_s L_{s-} | \mathcal{G}_s^I] d\hat{\xi}_s \\ \mathbb{E}^{\hat{\mathbb{P}}} \left[\int_0^t \phi_s L_{s-} ds | \mathcal{G}_t^I \right] &= \int_0^t \mathbb{E}^{\hat{\mathbb{P}}} [\phi_s L_{s-} | \mathcal{G}_s^I] ds \end{aligned}$$

Observing that, under $\hat{\mathbb{P}}$, $dY_t = \Sigma_Y d\hat{W}_t$, using that $Q(t, e_i, H_t, \pi_t)$ and $\eta(t, e_i, H_t, \pi_t)$ are $(\mathcal{G}_t^I)_{t \geq 0}$ adapted, and that the Markov-chain generator $A(t)$ is deterministic, we obtain Eq. (19), upon taking the differential of Eq. (67). \square

Proof of Proposition 3.2.

Using Itô's formula on Eq. (21), we have

$$dp_t^i = \frac{1}{M_{t-}} dq_t^i + q_{t-}^i d \left(\frac{1}{M_t} \right) + \left[\frac{1}{M}, q \right]_t. \quad (68)$$

First, using that $\sum_{i=1}^N \sum_{\ell=1}^N \varpi_{\ell,i}(t) q_t^\ell dt = \sum_{\ell=1}^N q_t^\ell \sum_{i=1}^N \varpi_{\ell,i}(t) dt = 0$ and Lemma 3.1, we obtain

$$dM_t = \sum_{i=1}^N q_t^i Q'(t, e_i, H_t, \pi_t) \Sigma_Y d\hat{W}_t + \sum_{i=1}^N q_{t-}^i (h_i - 1) d\hat{\xi}_t - \gamma \sum_{i=1}^N \eta(t, e_i, H_t, \pi_t) q_t^i dt. \quad (69)$$

Next, let us derive the dynamics of $\frac{1}{M_t}$. Using Itô's formula on the function $f(M_t) = \frac{1}{M_t}$, we have

$$f(M_t) = 1 + \int_0^t f'(M_{s-}) dM_s + \frac{1}{2} \int_0^t f''(M_{s-}) d\langle M^c, M^c \rangle_s + \sum_{0 < s \leq t} (f(M_s) - f(M_{s-}) - f'(M_{s-}) \Delta M_s). \quad (70)$$

Here, $d\langle M^c, M^c \rangle_s$ is the quadratic variation of the continuous part of M , which, in light of (69) and the definition of filter probability in Eq. (21), is such that

$$\begin{aligned} f''(M_{t-}) d\langle M^c, M^c \rangle_t &= \frac{1}{M_{t-}^3} \left(\sum_{j=1}^N \sum_{k=1}^N q_t^j Q'(t, e_j, H_t, \pi_t) \Sigma_Y \Sigma_Y' Q(t, e_k, H_t, \pi_t) q_t^k \right) dt \\ &= \frac{1}{M_{t-}} \sum_{j=1}^N \sum_{k=1}^N p_t^j Q'(t, e_j, H_t, \pi_t) \Sigma_Y \Sigma_Y' Q(t, e_k, H_t, \pi_t) p_t^k dt. \end{aligned} \quad (71)$$

Next, from Eqs. (21) and (69), note that

$$\frac{\Delta M_t}{M_{t-}} = \Delta H_t \frac{\sum_{i=1}^N q_{t-}^i (h_i - 1)}{M_{t-}} = \Delta H_t (\hat{h}_{t-} - 1), \quad (72)$$

where we have used the definition in (22), along with the obvious fact that $\sum_{i=1}^N p_{t-}^i = 1$. Therefore, $M_t/M_{t-} = 1 + \Delta H_t (\hat{h}_{t-} - 1)$ and, thus,

$$\frac{\Delta M_t}{M_t} = \frac{\Delta H_t (\hat{h}_{t-} - 1)}{1 + \Delta H_t (\hat{h}_{t-} - 1)} = \Delta H_t \frac{\hat{h}_{t-} - 1}{\hat{h}_{t-}}. \quad (73)$$

We then have

$$\begin{aligned} f(M_t) - f(M_{t-}) &= \Delta H_t \left(\frac{1}{M_t} - \frac{1}{M_{t-}} \right) = -\Delta H_t \frac{\Delta M_t}{M_{t-} M_t} \\ &= -\Delta H_t \frac{1}{M_{t-}} \frac{\hat{h}_{t-} - 1}{\hat{h}_{t-}}, \end{aligned} \quad (74)$$

where the last step follows using Eq. (73). Similarly, using Eq. (72), we have

$$\begin{aligned} f'(M_{t-}) \Delta M_t &= -\frac{1}{M_{t-}^2} \Delta M_t = -\frac{1}{M_{t-}} \frac{\Delta M_t}{M_{t-}} \\ &= -\Delta H_t \frac{\hat{h}_{t-} - 1}{M_{t-}}. \end{aligned} \quad (75)$$

Plugging Eq. (74), (75), and (71) into (70), we obtain

$$\begin{aligned} d\left(\frac{1}{M_t}\right) &= -\frac{1}{M_{t-}^2} \left(\sum_{i=1}^N q_t^i Q'(t, e_i, H_t, \pi_t) \Sigma_Y d\hat{W}_t - \gamma \sum_{i=1}^N \eta(t, e_i, H_t, \pi_t) q_t^i dt + \sum_{i=1}^N q_{t-}^i (h_i - 1) d\hat{\xi}_t \right) \\ &+ \frac{1}{M_{t-}^3} \left(\sum_{i=1}^N \sum_{j=1}^N q_t^i Q'(t, e_i, H_t, \pi_t) \Sigma_Y \Sigma_Y' Q(t, e_j, H_t, \pi_t) q_t^j \right) dt - \Delta H_t \left(\frac{\hat{h}_{t-} - 1}{\hat{h}_{t-} M_{t-}} - \frac{\hat{h}_{t-} - 1}{M_{t-}} \right) \\ &= -\frac{1}{M_{t-}} \left(\sum_{i=1}^N p_t^i Q'(t, e_i, H_t, \pi_t) \Sigma_Y d\hat{W}_t - \gamma \sum_{i=1}^N p_t^i \eta(t, e_i, H_t, \pi_t) dt \right) \\ &+ \frac{1}{M_{t-}} \sum_{i=1}^N \sum_{j=1}^N p_t^i Q'(t, e_i, H_t, \pi_t) \Sigma_Y \Sigma_Y' Q(t, e_j, H_t, \pi_t) p_t^j dt - \frac{1}{M_{t-}} \frac{\hat{h}_{t-} - 1}{\hat{h}_{t-}} \left(\Delta H_t - \hat{h}_{t-} \bar{H}(t^-) dt \right) \end{aligned}$$

where the last step follows from the observation that $\frac{1}{M_{t-}} \sum_{i=1}^N q_{t-}^i (h_i - 1) = \hat{h}_{t-} - 1$, and the definition of $\hat{\xi}_t$ in (16). Next, we have that the quadratic covariation between $\frac{1}{M_t}$ and q_t is given by

$$d \left[\frac{1}{M}, q \right]_t = -p_t^i Q'(t, e_i, H_t, \pi_t) \Sigma_Y \Sigma_Y' \left(\sum_j Q(t, e_j, H_t, \pi_t) p_t^j \right) dt + \Delta \left(\frac{1}{M_t} \right) \Delta q_t^i,$$

where, the dynamics of $1/M_t$ and q_t^i as well as (21) imply

$$\Delta \left(\frac{1}{M_t} \right) \Delta q_t^i = -\frac{\hat{h}_{t-} - 1}{\hat{h}_{t-}} p_{t-}^i (h_i - 1) \Delta H_t. \quad (76)$$

Plugging all terms into Eq. (68), we obtain

$$\begin{aligned} dp_t^i &= \sum_{\ell=1}^N \varpi_{\ell,i}(t) p_t^\ell dt + p_t^i (Q'(t, e_i, H_t, \pi_t) - Q'(t, p_t, H_t, \pi_t)) \Sigma_Y d\hat{W}_t + p_{t-}^i (h_i - 1) (\Delta H_t - \bar{H}(t^-) dt) \\ &- \gamma p_t^i (\eta(t, e_i, H_t, \pi_t) - \eta(t, p_t, H_t, \pi_t)) dt + p_t^i (Q'(t, p_t, H_t, \pi_t) - Q'(t, e_i, H_t, \pi_t)) \Sigma_Y \Sigma_Y' Q(t, p_t, H_t, \pi_t) dt \\ &- \frac{\hat{h}_{t-} - 1}{\hat{h}_{t-}} p_{t-}^i (\Delta H_t - \hat{h}_{t-} \bar{H}(t^-) dt) - \frac{\hat{h}_{t-} - 1}{\hat{h}_{t-}} p_{t-}^i (h_i - 1) \Delta H_t \end{aligned} \quad (77)$$

It can be checked directly that the jump component of (77) simplifies as follows:

$$p_{t-}^i (h_i - 1) (\Delta H_t - \bar{H}(t^-) dt) - \frac{\hat{h}_{t-} - 1}{\hat{h}_{t-}} p_{t-}^i (h_i \Delta H_t - \hat{h}_{t-} \bar{H}(t^-) dt) = \frac{p_{t-}^i (h_i - \hat{h}_{t-})}{\hat{h}_{t-}} (\Delta H_t - \hat{h}_{t-} \bar{H}(t^-) dt),$$

which in turn allow us to rewrite Eq. (77) as

$$\begin{aligned} dp_t^i &= \sum_{\ell=1}^N \varpi_{\ell,i}(t) p_t^\ell dt + p_t^i (Q'(t, e_i, H_t, \pi_t) - Q'(t, p_t, H_t, \pi_t)) \Sigma_Y (d\hat{W}_t - \Sigma_Y' Q(t, p_t, H_t, \pi_t) dt) \\ &- \gamma p_t^i (\eta(t, e_i, H_t, \pi_t) - \eta(t, p_t, H_t, \pi_t)) dt + p_t^i \frac{h_i - \hat{h}_{t-}}{\hat{h}_{t-}} (\Delta H_t - \hat{h}_{t-} \bar{H}(t^-) dt). \end{aligned} \quad (78)$$

Next, from (10), (15), and (18), observe that

$$\begin{aligned} Q'(t, e_i, H_t, \pi_t) - Q'(t, p_t, H_t, \pi_t) &= (\vartheta(t, e_i, H_t)' - \vartheta(t, p_t, H_t)') (\Sigma_Y \Sigma_Y')^{-1}, \\ \Sigma_Y \Sigma_Y' Q(t, p_t, H_t, \pi_t) &= \vartheta(t, p_t, H_t) + \gamma \Sigma_Y \Sigma_Y' \pi_t, \\ \eta(t, e_i, H_t, \pi_t) - \eta(t, p_t, H_t, \pi_t) &= \pi_t' (\vartheta(t, p_t, H_t) - \vartheta(t, e_i, H_t)). \end{aligned}$$

Using the above relations, we obtain

$$\begin{aligned} &- p_t^i (Q'(t, e_i, H_t, \pi_t) - Q'(t, p_t, H_t, \pi_t)) \Sigma_Y \Sigma_Y' Q(t, p_t, H_t, \pi_t) dt - \gamma p_t^i (\eta(t, e_i, H_t, \pi_t) - \eta(t, p_t, H_t, \pi_t)) dt \\ &= -p_t^i (\vartheta'(t, e_i, H_t) - \vartheta'(t, p_t, H_t)) (\Sigma_Y \Sigma_Y')^{-1} \vartheta(t, p_t, H_t) dt \end{aligned}$$

Using the above equation, along with the fact that $dY_t = \Sigma_Y d\hat{W}_t$, we can simplify the dynamics in (78) to Eq. (23). \square

Proof of Lemma 3.3.

Define $\varsigma = \inf\{t : p_t^i = 0\} \wedge T$. If p_t^i can hit zero, then $\mathbb{P}(p_\varsigma^i = 0) > 0$. But $p_\varsigma^i = \frac{q_\varsigma^i}{\sum_j q_\varsigma^j}$, where the equality

$$q_\varsigma^i = \mathbb{E}^{\hat{\mathbb{P}}} [L_\varsigma \mathbf{1}_{X_\varsigma = e_i} | \mathcal{G}_\varsigma^I]$$

is true by the optional projection property, see Rogers and Williams (2006). Define the two dimensional (observed) log-price process $Y_t = (\log(S_t), \log(P_t))'$. As $q_\varsigma^i = \mathbb{E}^{\hat{\mathbb{P}}} [L_\varsigma \mathbf{1}_{X_\varsigma=e_i} | \mathcal{G}_\varsigma^I]$, and using that $L_\varsigma > 0$, we can choose a modification $g(Y, H, X_\varsigma)$ of $\mathbb{E}^{\hat{\mathbb{P}}} [L_\varsigma | \mathcal{G}_\varsigma^I, X_\varsigma]$ such that $g > 0$, and, for each e_i , $g(Y, H, e_i)$ is \mathcal{G}_ς^I -measurable. By the tower property

$$q_\varsigma^i = \mathbb{E}^{\hat{\mathbb{P}}} [g(Y, H, X_\varsigma) \mathbf{1}_{X_\varsigma=e_i} | \mathcal{G}_\varsigma^I] = g(Y, H, e_i) \hat{\mathbb{P}}(X_\varsigma = e_i | \mathcal{G}_\varsigma^I) = g(Y, H, e_i) \mathbb{P}(X_t = e_i) \big|_{t=\varsigma},$$

where the first equality follows because ς is \mathcal{G}_ς^I -measurable and the last two equalities because X is independent of \mathcal{G}^I under $\hat{\mathbb{P}}$. As $\mathbb{P}(X_t = e_i) > 0$ and $g > 0$, we get that $q_\varsigma^i > 0$ a.s, which contradicts that $\mathbb{P}(p_\varsigma^i = 0) > 0$. \square

Proof of Proposition 3.4.

We first establish the following relation

$$q_t^i = \hat{L}_t p_t^i. \quad (79)$$

We have

$$d(\hat{L}_t p_t^i) = \hat{L}_t - dp_t^i + p_t^i d\hat{L}_t + d \left[\hat{L}, p^i \right]_t.$$

In light of Eq. (11) and (12), we have that

$$d\hat{L}_t = \hat{L}_{t-} \left(Q'(t, p_t, H_t, \pi_t) dY_t + (\hat{h}_{t-} - 1) d\hat{\xi}_t \right) - \gamma \eta(t, p_t, H_t, \pi_t) \hat{L}_t dt. \quad (80)$$

From Eq. (80) and (23), we obtain

$$\begin{aligned} d \left[\hat{L}, p^i \right]_t &= p_t^i \hat{L}_t \vartheta'(t, p_t, H_t) \Sigma_Y \Sigma_Y^{-1} (\vartheta(t, e_i, H_t) - \vartheta(t, p_t, H_t)) dt + p_t^i \hat{L}_t \gamma \pi_t' (\vartheta(t, e_i, H_t) - \vartheta(t, p_t, H_t)) dt \\ &\quad + \left(\hat{h}_{t-} - 1 \right) \frac{h_i - \hat{h}_{t-}}{\hat{h}_{t-}} \hat{L}_{t-} p_{t-}^i \Delta H_t. \end{aligned} \quad (81)$$

Using the above equations, along with (23), we obtain

$$\begin{aligned} d \left(\hat{L}_t p_t^i \right) &= \hat{L}_t \left(\sum_{\ell=1}^N \varpi_{\ell,i}(t) p_t^\ell dt \right) + \hat{L}_t p_t^i (\vartheta'(t, e_i, H_t) - \vartheta'(t, p_t, H_t)) (\Sigma_Y \Sigma_Y')^{-1} (dY_t - \vartheta(t, p_t, H_t) dt) \\ &\quad + \hat{L}_{t-} p_{t-}^i \frac{h_i - \hat{h}_{t-}}{\hat{h}_{t-}} \left(\Delta H_t - \hat{h}_{t-} \bar{H}(t^-) dt \right) + p_t^i \hat{L}_t Q'(t, p_t, H_t, \pi_t) dY_t - p_t^i \hat{L}_t \gamma \eta(t, p_t, H_t, \pi_t) dt \\ &\quad + p_{t-}^i \hat{L}_{t-} (\hat{h}_{t-} - 1) (\Delta H_t - \bar{H}(t^-) dt) + (\hat{h}_{t-} - 1) \frac{h_i - \hat{h}_{t-}}{\hat{h}_{t-}} \hat{L}_{t-} p_{t-}^i \Delta H_t \\ &\quad + p_t^i \hat{L}_t \vartheta'(t, p_t, H_t) (\Sigma_Y \Sigma_Y')^{-1} (\vartheta(t, e_i, H_t) - \vartheta(t, p_t, H_t)) dt + \gamma p_t^i \hat{L}_t \pi_t' (\vartheta(t, e_i, H_t) - \vartheta(t, p_t, H_t)) dt. \end{aligned} \quad (82)$$

Next, observe that

$$\begin{aligned} \hat{L}_t p_t^i (\vartheta'(t, e_i, H_t) - \vartheta'(t, p_t, H_t)) (\Sigma_Y \Sigma_Y')^{-1} (dY_t - \vartheta(t, p_t, H_t) dt) + p_t^i \hat{L}_t Q'(t, p_t, H_t, \pi_t) dY_t &= \\ \hat{L}_t p_t^i Q'(t, e_i, H_t, \pi_t) dY_t - \hat{L}_t p_t^i (\vartheta'(t, e_i, H_t) - \vartheta'(t, p_t, H_t)) (\Sigma_Y \Sigma_Y')^{-1} \vartheta(t, p_t, H_t) dt & \end{aligned} \quad (83)$$

Moreover,

$$\eta(t, e_i, H_t, \pi_t) - \eta(t, p_t, H_t, \pi_t) = \pi_t' (\vartheta(t, p_t, H_t) - \vartheta(t, e_i, H_t)). \quad (84)$$

Using relations (83), and (84), along with straightforward simplifications, we may simplify Eq. (82) to

$$d(\hat{L}_t p_t^i) = \left(\sum_{\ell=1}^N \varpi_{\ell,i}(t) \hat{L}_t p_t^\ell dt \right) + \hat{L}_t p_t^i Q'(t, e_i, H_t, \pi_t) dY_t - \gamma \hat{L}_t p_t^i \eta(t, e_i, H_t, \pi_t) dt + \hat{L}_{t-} p_{t-}^i (h_i - 1) d\hat{\xi}_t. \quad (85)$$

Using that $dY_t = \Sigma_Y d\hat{W}_t$, we have that the equality (79) holds via a direct comparison of equations (85) and (19). Next, using Eq. (17), and the established relation (79), we have that

$$\begin{aligned} J(v, \pi, T) &= \frac{1}{\gamma} \mathbb{E} [V_T^\gamma] = \frac{v^\gamma}{\gamma} \mathbb{E}^\mathbb{P} [L_T] = \frac{v^\gamma}{\gamma} \mathbb{E}^\mathbb{P} \left[\mathbb{E}^\mathbb{P} [L_T | \mathcal{G}_T^I] \right] \\ &= \frac{v^\gamma}{\gamma} \sum_{i=1}^N \mathbb{E}^\mathbb{P} \left[\mathbb{E}^\mathbb{P} [L_T \mathbf{1}_{\{X_T = e_i\}} | \mathcal{G}_T^I] \right] = \frac{v^\gamma}{\gamma} \sum_{i=1}^N \mathbb{E}^\mathbb{P} [q_T^i] \\ &= \frac{v^\gamma}{\gamma} \sum_{i=1}^N \mathbb{E}^\mathbb{P} [\hat{L}_T p_T^i] = \frac{v^\gamma}{\gamma} \mathbb{E}^\mathbb{P} [\hat{L}_T], \end{aligned}$$

thus proving the statement. \square

B Proofs related to Section 5

Proof of Eq. (59)

Let us first analyze the first term in the sup of Eq. (57), i.e. $\bar{\beta}'_\gamma \nabla w$. For brevity, let us set $\bar{\beta} = \beta(t, p, 0)$. By definition of $\bar{\beta}_\gamma$, and using the maximizer $\pi_t := \pi_t^*$ in (58), we have

$$\begin{aligned} \bar{\beta}'_\gamma &= \tilde{\beta}' + \gamma \pi'_t \Sigma_Y \bar{\alpha}' &= \tilde{\beta}' + \frac{\gamma}{1-\gamma} (\Sigma_Y \bar{\alpha}' (\nabla_p \bar{w})' - \Upsilon)' (\Sigma'_Y \Sigma_Y)^{-1} \Sigma_Y \bar{\alpha}' \\ &= \tilde{\beta}' + \frac{\gamma}{1-\gamma} (\nabla_p \bar{w}) \bar{\alpha} \Sigma'_Y (\Sigma'_Y \Sigma_Y)^{-1} \Sigma_Y \bar{\alpha}' - \frac{\gamma}{1-\gamma} \Upsilon' (\Sigma'_Y \Sigma_Y)^{-1} \Sigma_Y \bar{\alpha}' \end{aligned}$$

Further, again using the expression for $\pi = \pi^*$, the second term in the sup is equal to

$$\begin{aligned} -\gamma \pi'_t \Upsilon &= -\frac{\gamma}{1-\gamma} (-\Upsilon + \Sigma_Y \bar{\alpha}' (\nabla_p \bar{w})')' (\Sigma'_Y \Sigma_Y)^{-1} \Upsilon \\ &= \frac{\gamma}{1-\gamma} \Upsilon' (\Sigma'_Y \Sigma_Y)^{-1} \Upsilon - \frac{\gamma}{1-\gamma} (\nabla_p \bar{w}) \bar{\alpha} \Sigma'_Y (\Sigma'_Y \Sigma_Y)^{-1} \Upsilon. \end{aligned} \quad (86)$$

The third term in the sup may be simplified as

$$\begin{aligned} &-\frac{1}{2} \frac{\gamma}{1-\gamma} (-\Upsilon + \Sigma_Y \bar{\alpha}' (\nabla_p \bar{w})')' (\Sigma'_Y \Sigma_Y)^{-1} (-\Upsilon + \Sigma_Y \bar{\alpha}' (\nabla_p \bar{w})') = \\ &-\frac{1}{2} \frac{\gamma}{1-\gamma} \Upsilon' (\Sigma'_Y \Sigma_Y)^{-1} \Upsilon + \frac{1}{2} \frac{\gamma}{1-\gamma} \Upsilon' (\Sigma'_Y \Sigma_Y)^{-1} \Sigma_Y \bar{\alpha}' (\nabla_p \bar{w})' \\ &+ \frac{1}{2} \frac{\gamma}{1-\gamma} (\nabla_p \bar{w}) \bar{\alpha} \Sigma'_Y (\Sigma'_Y \Sigma_Y)^{-1} \Upsilon - \frac{1}{2} \frac{\gamma}{1-\gamma} (\nabla_p \bar{w}) \bar{\alpha} \Sigma'_Y (\Sigma'_Y \Sigma_Y)^{-1} \Sigma_Y \bar{\alpha}' (\nabla_p \bar{w})'. \end{aligned} \quad (87)$$

Using Eq. (86), (86), and (87), we obtain that

$$\begin{aligned} \sup_{\pi} \left\{ \bar{\beta}'_\gamma (\nabla_p \bar{w})' - \gamma \pi'_t \Upsilon - \frac{1}{2} \gamma (1-\gamma) \pi'_t \Sigma'_Y \Sigma_Y \pi_t \right\} &= \\ \tilde{\beta}' (\nabla_p \bar{w})' + \frac{1}{2} \frac{\gamma}{1-\gamma} (\nabla_p \bar{w}) \bar{\alpha} \Sigma'_Y (\Sigma'_Y \Sigma_Y)^{-1} \Sigma_Y \bar{\alpha}' (\nabla_p \bar{w})' &+ \frac{1}{2} \frac{\gamma}{1-\gamma} \Upsilon' (\Sigma'_Y \Sigma_Y)^{-1} \Upsilon - \frac{\gamma}{1-\gamma} (\nabla_p \bar{w}) \bar{\alpha} \Sigma_Y^{-1} \Upsilon, \end{aligned}$$

and therefore, after re-arrangement, we obtain Eq. (59). \square

Proof of Theorem 5.1

The first assertion of the theorem follows from Feynman-Kac formula as used in, e.g., Tamura and Watanabe (2011) (see proof of Theorem 3.1 therein). Indeed, the idea (see also Nagai and Runggaldier (2008)) is to transform the

problem into a linear PDE. Concretely, defining two open (bounded) balls $B_2 \supseteq B_1 \supseteq \tilde{\Delta}_{N-1}$ and a truncation function $\tau \in C^\infty$ such that $\mathbf{1}_{B_1}(\tilde{p}) \leq \tau(\tilde{p}) \leq \mathbf{1}_{B_2}(\tilde{p})$, define the function

$$\bar{\psi}(t, \tilde{p}) = \mathbb{E} \left(e^{\frac{1}{1-\gamma} \int_0^{T-t} \tau(\underline{X}_s^{\tilde{p}}) \underline{\Psi}(s, \underline{X}_s^{\tilde{p}}) ds} \right), \quad (88)$$

where $\{\underline{X}_s^x\}_{0 \leq s \leq T}$ is the solution of the SDE:

$$d\underline{X}_s^x = \tau(\underline{X}_s^x) \underline{\alpha}(\underline{X}_s^x) dB_s + \tau(\underline{X}_s^x) \underline{\Phi}(\underline{X}_s^x) ds, \quad X_0^x = x,$$

with B denoting an $N-1$ -dimensional Wiener process. Then, Feynman-Kac implies that the function (88) solves the PDE

$$\underline{\psi}_t + \frac{\tau^2(\tilde{p})}{2} \text{tr}(\underline{\alpha} \underline{\alpha}' D^2 \underline{\psi}) + \tau(\tilde{p}) (\nabla_{\tilde{p}} \underline{\psi}) \underline{\Phi} + \tau(\tilde{p}) \underline{\psi}(t, \tilde{p}) \underline{\Psi} = 0, \quad \underline{\psi}(T, \tilde{p}) = 0. \quad (89)$$

Next, it is easy to check that

$$\underline{w}(t, \tilde{p}) := (1 - \alpha) \log(\underline{\psi}(t, \tilde{p}))$$

satisfies the PDE

$$\underline{w}_t + \frac{\tau^2(\tilde{p})}{2} \text{tr}(\underline{\alpha} \underline{\alpha}' D^2 \underline{w}) + \frac{\tau^2(\tilde{p})}{2(1-\gamma)} (\nabla_{\tilde{p}} \underline{w}) \underline{\alpha} \underline{\alpha}' (\nabla_{\tilde{p}} \underline{w})' + \tau(\tilde{p}) (\nabla_{\tilde{p}} \underline{w}) \underline{\Phi} + \tau(\tilde{p}) \underline{\Psi} = 0, \quad (90)$$

with the boundary condition $\underline{w}(T, \tilde{p}) = 0$. This suffices to conclude the validity of the first assertion since $\tau(\tilde{p}) = 1$ on $\tilde{\Delta}_{N-1}$ by construction.

We now proceed to prove the last two assertions of the theorem. Hereafter, we set $\pi_s^P \equiv 0$. Fix a feedback control $\pi_s^S := \pi^S(s, \tilde{p}_s)$ such that $(\pi^S, \pi^P) \in \bar{\mathcal{A}}(t, T; \tilde{p}, 1)$. Let us first remark that

$$\tilde{\mathbb{P}}(\tilde{p}_s \in \tilde{\Delta}_{N-1}^\circ, t \leq s \leq T) = 1. \quad (91)$$

or, equivalently, $\mathbb{P}(\tilde{p}_s \in \tilde{\Delta}_{N-1}^\circ, t \leq s \leq T) = 1$. Indeed, set $\tilde{p}_s^N = 1 - \sum_{j=1}^{N-1} \tilde{p}_s^j$ and note that the process \tilde{p}_s^i can be represented as

$$\tilde{p}_s^i := \mathbb{P}_t(X_s = e_i | \mathcal{G}_s^I) = \frac{1}{M_s} \mathbb{E}^{\tilde{\mathbb{P}}_t} \left[L_s \mathbf{1}_{\{X_s = e_i\}} \middle| \mathcal{G}_s^I \right], \quad (s \geq t, i = 1, \dots, N), \quad (92)$$

where $M_s := \sum_{j=1}^N \mathbb{E}_t^{\tilde{\mathbb{P}}}[L_s \mathbf{1}_{\{X_s = e_j\}} | \mathcal{G}_s^I]$. The previous representation is the analogous of (21) but starting at time t instead of time 0. Above, \mathbb{P}_t represents a probability measure such that $\mathbb{P}_t(X_t = e_i) = \tilde{p}_t^i$ and $\mathbb{E}^{\tilde{\mathbb{P}}_t}$ represents the expectation with respect to a probability measure $\tilde{\mathbb{P}}_t$, which is constructed from \mathbb{P}_t in the same way as $\tilde{\mathbb{P}}$ is constructed from \mathbb{P} . From the representation (92), it is clear that $\tilde{p}_s = (\tilde{p}_s^1, \dots, \tilde{p}_s^{N-1})' \in \tilde{\Delta}_{N-1}$. Furthermore, similarly to Lemma 3.3, all the \tilde{p}_s^i , with $i = 1, \dots, N$, remain positive in $[t, T]$, a.s., and, hence, (91) is satisfied.

Next, for each feedback control $\pi_s^S := \pi^S(s, \tilde{p}_s)$ such that $(\pi^S, \pi^P) \in \bar{\mathcal{A}}(t, T; \tilde{p}, 1)$, define the process

$$M_s^{\pi^S} := e^{-\gamma \int_t^s \underline{\eta}(u, \tilde{p}_u, \pi_u^S) du} e^{\underline{w}(s, \tilde{p}_s)}, \quad (t \leq s \leq T), \quad (93)$$

where

$$\underline{\eta}(t, \tilde{p}, \pi) = \tilde{\eta}(t, \tilde{p}, 1, (\pi, 0)') = -r + \pi(r - \tilde{\mu}(\tilde{p})) + \frac{1-\gamma}{2} \sigma^2 \pi^2. \quad (94)$$

In what follows, we write for simplicity M^π for M^{π^S} . Note that the process $\{M_s^\pi\}_{t \leq s \leq T}$ is uniformly bounded. Indeed, (94) is convex in π and by minimizing it over π , it follows that, for any $\tilde{p} \in \tilde{\Delta}_{N-1}$,

$$-\underline{\eta}(t, \tilde{p}, \pi) \leq r + \frac{(\tilde{\mu}(\tilde{p}) - r)^2}{2(1-\gamma)\sigma^2} \leq r + \frac{(\max_i \mu_i^2 + r^2)}{(1-\gamma)\sigma^2} < \infty.$$

Therefore, since $\underline{w} \in C([0, T] \times \tilde{\Delta}_{N-1})$, there exists a constant $K < \infty$ for which

$$M_s^\pi = e^{-\gamma \int_t^s \underline{\eta}(u, \tilde{p}_u, \pi_u) du} e^{\underline{w}(s, \tilde{p}_s)} \leq K e^{\gamma \|\underline{\eta}\|_\infty (T-t)} =: A < \infty. \quad (95)$$

We prove the result through the following steps:

(i) Define the process $\mathcal{Y}_s = e^{\underline{w}(s, \tilde{p}_s)}$. By Itô's formula and the generator formula (36) with $f(s, \tilde{p}) = e^{\underline{w}(s, \tilde{p})}$,

$$\begin{aligned} M_s^\pi &= M_t^\pi + \int_t^s e^{-\gamma \int_t^u \underline{\eta}(r, \tilde{p}_r, \pi_r) dr} d\mathcal{Y}_u - \gamma \int_t^s \underline{\eta}(u, \tilde{p}_u, \pi_u) e^{-\gamma \int_t^u \underline{\eta}(r, \tilde{p}_r, \pi_r) dr} \mathcal{Y}_u du \\ &= M_t^\pi + \int_t^s M_u^\pi \left(\frac{\partial \underline{w}}{\partial u} + \frac{1}{2} \text{tr}(\underline{\alpha} \underline{\alpha}' D^2 \underline{w}) + \frac{1}{2} (\nabla_{\tilde{p}} \underline{w}) \underline{\alpha} \underline{\alpha}' (\nabla_{\tilde{p}} \underline{w})' + (\nabla_{\tilde{p}} \underline{w}) \underline{\beta}_\gamma - \gamma \underline{\eta} \right) du + \int_t^s M_u^\pi \nabla_{\tilde{p}} \underline{w} \underline{\alpha} d\tilde{W}_u^{(1)}. \end{aligned}$$

Using the expression of $\underline{\eta}$ in (94) and some rearrangement, we may write M^π as

$$M_s^\pi = M_t^\pi + \int_t^s M_u^\pi R(u, \tilde{p}_u, \pi_u) du + \int_t^s M_u^\pi \nabla_{\tilde{p}} \underline{w} \underline{\alpha} d\tilde{W}_u^{(1)}$$

with

$$R(u, \tilde{p}, \pi) = \underline{w}_u + \frac{1}{2} \text{tr}(\underline{\alpha} \underline{\alpha}' D^2 \underline{w}) + \frac{1}{2} (\nabla_{\tilde{p}} \underline{w}) \underline{\alpha} \underline{\alpha}' (\nabla_{\tilde{p}} \underline{w})' + \gamma r + (\nabla_{\tilde{p}} \underline{w}) \underline{\beta}_\gamma - \gamma \pi (r - \tilde{\mu}(\tilde{p})) - \frac{\sigma^2}{2} \gamma (1 - \gamma) \pi^2. \quad (96)$$

Clearly, $R(u, \tilde{p}, \pi)$ is a concave function in π since $R_{\pi\pi} = -\sigma^2 \gamma (1 - \gamma) < 0$. If we maximize $R(u, \tilde{p}, \pi)$ as a function of π for each (u, \tilde{p}) , we find that the optimum is given by (53). Upon substituting (53) into (96), we get that

$$R(u, \tilde{p}, \pi) \leq R(u, \tilde{p}, \tilde{\pi}^S(u, \tilde{p})) = \underline{w}_u + \frac{1}{2} \text{tr}(\underline{\alpha} \underline{\alpha}' D^2 \underline{w}) + \frac{1}{2(1-\gamma)} (\nabla_{\tilde{p}} \underline{w}) \underline{\alpha} \underline{\alpha}' (\nabla_{\tilde{p}} \underline{w})' + (\nabla_{\tilde{p}} \underline{w}) \underline{\Phi} + \underline{\Psi} = 0,$$

where the last equality follows from Eq. (51). Next, let us introduce the stopping time

$$\tau_a := \inf \left\{ u \geq t : \sum_{i=1}^{N-1} \tilde{p}_u^i > 1 - a \text{ or } \min_{1 \leq i \leq N-1} \{\tilde{p}_u^i\} < a \right\},$$

for a small enough $a > 0$. Note that $(\tau_a \wedge T) \nearrow T$ as $a \searrow 0$ since $\tilde{\mathbb{P}}(\tilde{p}_s \in \tilde{\Delta}_{N-1}^\circ, t \leq s \leq T) = 1$. Then, using the shorthand notation $\mathbb{E}_t^{\tilde{\mathbb{P}}}[\cdot] = \mathbb{E}_{t, \tilde{p}, 1}^{\tilde{\mathbb{P}}}[\cdot]$, we get the inequality

$$\mathbb{E}_t^{\tilde{\mathbb{P}}} [M_{s \wedge \tau_a}^\pi] \leq M_t^\pi + \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[\int_t^{s \wedge \tau_a} M_u^\pi \nabla_{\tilde{p}} \underline{w} \underline{\alpha} d\tilde{W}_u^{(1)} \right],$$

with equality if $\pi = \tilde{\pi}^S$. Next, from (49), it is easy to check that $\sup_{\tilde{p} \in \tilde{\Delta}_{N-1}} \|\underline{\alpha}(\tilde{p})\|^2 \leq 2 \max_i \{\mu_i\} / \sigma$. Then, since $\underline{w} \in C^{1,2}((0, T) \times \tilde{\Delta}_{N-1}^\circ) \cap C([0, T] \times \tilde{\Delta}_{N-1})$, we have

$$\sup_{t \leq u \leq \tau_a \wedge T} |M_u^\pi \nabla_{\tilde{p}} \underline{w} \underline{\alpha}|^2 \leq A \sup_{t \leq u \leq T} \|\underline{\alpha}(\tilde{p}_u)\|^2 \sup_{t \leq u \leq \tau_a \wedge T} \|\nabla_{\tilde{p}} \underline{w}(u, \tilde{p}_u)\|^2 \leq B,$$

for some constant $B < \infty$. We conclude that

$$\mathbb{E}_t^{\tilde{\mathbb{P}}} [M_{T \wedge \tau_a}^\pi] \leq M_t^\pi = e^{\underline{w}(t, \tilde{p}_t)} = e^{\underline{w}(t, \tilde{p})}, \quad (97)$$

with equality if $\pi = \tilde{\pi}^S$.

(ii) For simplicity, let us write $\tilde{\pi}_s := \tilde{\pi}^S(s, \tilde{p}_s)$. First note that by (95) and the Dominated Convergence Theorem,

$$\lim_{a \rightarrow 0} \mathbb{E}_t^{\tilde{\mathbb{P}}} [M_{T \wedge \tau_a}^{\tilde{\pi}}] = \mathbb{E}_t^{\tilde{\mathbb{P}}} [M_T^{\tilde{\pi}}], \quad (98)$$

since $(\tau_a \wedge T) \nearrow T$ as $a \searrow 0$. From (98) and the fact that we have equality in (97) when $\pi = \tilde{\pi}$,

$$e^{\underline{w}(t, \tilde{p})} = \lim_{a \rightarrow 0} \mathbb{E}_t^{\tilde{\mathbb{P}}} [M_{T \wedge \tau_a}^{\tilde{\pi}}] = \mathbb{E}_t^{\tilde{\mathbb{P}}} [M_T^{\tilde{\pi}}] = \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[e^{-\gamma \int_t^T \underline{\eta}(u, \tilde{p}_u, \tilde{\pi}_u) du} e^{\underline{w}(T, \tilde{p}_T)} \right] = \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[e^{-\gamma \int_t^T \underline{\eta}(u, \tilde{p}_u, \tilde{\pi}_u) du} \right]. \quad (99)$$

Similarly, from (95) and dominated convergence theorem, for every feedback control $\pi_s = \pi(s, \tilde{p}_s)$ such that $(\pi, 0) \in \mathcal{A}(t, T; \tilde{p}, 1)$,

$$\mathbb{E}_t^{\tilde{\mathbb{P}}} \left[e^{-\gamma \int_t^T \underline{\eta}(u, \tilde{p}_u, \pi_u) du} \right] = \mathbb{E}_t^{\tilde{\mathbb{P}}} [M_T^\pi] = \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[\lim_{a \rightarrow 0} M_{T \wedge \tau_a}^\pi \right] = \lim_{a \rightarrow 0} \mathbb{E}_t^{\tilde{\mathbb{P}}} [M_{T \wedge \tau_a}^\pi] \leq M_t^\pi = e^{\underline{w}(t, \tilde{p})} = \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[e^{-\gamma \int_t^T \underline{\eta}(u, \tilde{p}_u, \tilde{\pi}_u) du} \right],$$

where the inequality in the previous equation comes from (97) and the last equality therein follows from (99). The previous relationships show the optimality of $\tilde{\pi}$ and prove the assertions (1) and (2). \square

Proof of Theorem 5.2

As in the proof of the post default verification theorem, let us first note that for any feedback control $\pi_s := (\pi_s^S, \pi_s^P) := (\pi^S(s, \tilde{p}_{s-}, H_{s-}), \pi^P(s, \tilde{p}_{s-}, H_{s-}))$ such that $(\pi^S, \pi^P) \in \tilde{\mathcal{A}}(t, T; \tilde{p}, 0)$, we have

$$\tilde{\mathbb{P}}(\tilde{p}_s \in \tilde{\Delta}_{N-1}^\circ, t \leq s \leq T) = 1. \quad (100)$$

Next, define the process

$$M_s^\pi := e^{-\gamma \int_t^s \tilde{\eta}(u, \tilde{p}_u, H_u, \pi_u) du} e^{w(s, \tilde{p}_s, H_s)}, \quad (t \leq s \leq T), \quad (101)$$

where $w(s, \tilde{p}, z) := (1 - z)\bar{w}(s, \tilde{p}) + z\underline{w}(s, \tilde{p})$ and $\tilde{\eta}$ defined as in Eq. (45). Note that $\tilde{\eta}$ can be written as

$$\tilde{\eta}(t, \tilde{p}, z, \pi) = -r + \pi^S(r - \tilde{\mu}(\tilde{p})) + \frac{1 - \gamma}{2} \sigma^2 (\pi^S)^2 + \pi^P \left(r - \frac{1}{2} v^2 - \tilde{a}(t, \tilde{p}, z) \right) + \frac{1 - \gamma}{2} v^2 (\pi^P)^2,$$

and, thus, $-\tilde{\eta}$ is concave. This in turn implies that there exists a constant $A < \infty$ such that

$$0 < M_s^\pi \leq A < \infty, \quad t \leq s \leq T, \quad (102)$$

since $\underline{w}, \bar{w} \in C([0, T] \times \tilde{\Delta}_{N-1})$. We prove the result through the following steps:

(i) Define the processes $\mathcal{Y}_s = e^{w(s, \tilde{p}, H_s)}$ and $\mathcal{U}_s = e^{-\gamma \int_t^s \tilde{\eta}(u, \tilde{p}_u, H_u, \pi_u) du}$. By Itô's formula, the generator formula (36) with $f(s, \tilde{p}, z) = e^{w(s, \tilde{p}, z)}$, and the same arguments as those used to derive (43),

$$\begin{aligned} M_s^\pi &= M_t^\pi + \int_t^s \mathcal{U}_u d\mathcal{Y}_u - \gamma \int_t^s \tilde{\eta}(u, \tilde{p}_u, H_u, \pi_u) \mathcal{U}_u \mathcal{Y}_u du \\ &= M_t^\pi + \int_t^s M_u^\pi \left[\frac{\partial w}{\partial u} + \frac{1}{2} \text{tr}(\alpha \alpha' D^2 w) + \frac{1}{2} (\nabla_{\tilde{p}} w) \alpha \alpha' (\nabla_{\tilde{p}} w)' + (\nabla_{\tilde{p}} w) \beta_\gamma \right. \\ &\quad \left. + (1 - H_u) \tilde{h}(\tilde{p}_u) \left(e^{\underline{w}(u, \frac{1}{h(\tilde{p}_u)} \tilde{p}_u \cdot h^\perp)} - \bar{w}(u, \tilde{p}_u) - 1 \right) - \gamma \tilde{\eta} \right] du + \mathcal{M}_s^c + \mathcal{M}_s^d, \end{aligned}$$

where

$$\mathcal{M}_s^c := \int_t^s M_u^\pi \nabla_{\tilde{p}} w \alpha(u, \tilde{p}_u, H_u) d\tilde{W}_u, \quad \mathcal{M}_s^d := \int_t^s \mathcal{U}_u - \left(e^{\underline{w}(u, \frac{1}{h(\tilde{p}_u)} \tilde{p}_u \cdot h^\perp)} - e^{\bar{w}(u, \tilde{p}_u)} \right) d\tilde{\xi}_u. \quad (103)$$

Using the expression of η in Eq. (45), and similar arguments to those used to derive (45), we may write M^π as

$$M_s^\pi = M_t^\pi + \int_t^s M_u^\pi R(u, \tilde{p}_u, \pi_u, H_u) du + \mathcal{M}_s^c + \mathcal{M}_s^d$$

with

$$\begin{aligned} R(u, \tilde{p}, \pi, z) &= \frac{\partial w}{\partial u} + \frac{1}{2} \text{tr}(\alpha \alpha' D^2 w) + \frac{1}{2} (\nabla_{\tilde{p}} w) \alpha \alpha' (\nabla_{\tilde{p}} w)' + \gamma r + (1 - z) \tilde{h}(\tilde{p}) \left[e^{\underline{w}(t, \frac{1}{h(\tilde{p})} \tilde{p} \cdot h^\perp)} - \bar{w}(t, \tilde{p}) - 1 \right] \\ &\quad + z \left((\nabla_{\tilde{p}} \underline{w}) \beta_\gamma - \gamma \pi^S(r - \tilde{\mu}(\tilde{p})) - \frac{\gamma(1 - \gamma)}{2} \sigma^2 (\pi^S)^2 \right) \\ &\quad + (1 - z) \left((\nabla_{\tilde{p}} \bar{w}) \beta_\gamma - \gamma \pi^P(r - \frac{1}{2} v^2 - \tilde{a}(t, \tilde{p}, z)) - \frac{\gamma(1 - \gamma)}{2} v^2 (\pi^P)^2 \right) \end{aligned} \quad (104)$$

Clearly, $R(u, \tilde{p}, \pi, z)$ is a concave function in π for each (u, \tilde{p}, z) . Furthermore, this function reaches its maximum at $\tilde{\pi}(u, \tilde{p}, z) = (\tilde{\pi}^S(u, \tilde{p}, z), \tilde{\pi}^P(u, \tilde{p}, z))$ as defined in the statement of the theorem. Upon substituting this maximum into (104) and rearrangements similar to those leading to (51) and (59) (depending on whether $z = 1$ or $z = 0$), we get

$$R(u, \tilde{p}, \pi, z) \leq R(u, \tilde{p}, \tilde{\pi}(u, \tilde{p}, z), z) = 0,$$

in light of the corresponding equations (51) and (59). As in the post default problem, we introduce the stopping time

$$\tau_a := \inf \left\{ u \geq t : \sum_{i=1}^{N-1} \tilde{p}_u^i > 1 - a \text{ or } \min_{1 \leq i \leq N-1} \{\tilde{p}_u^i\} < a \right\},$$

for a small enough $a > 0$. Note that $(\tau_a \wedge T) \nearrow T$ as $a \searrow 0$ since $\tilde{\mathbb{P}}(\tilde{p}_s \in \tilde{\Delta}_{N-1}^\circ, t \leq s \leq T) = 1$.

Then, using the shorthand notation $\mathbb{E}_t^{\tilde{\mathbb{P}}}[\cdot] = \mathbb{E}_{t, \tilde{p}, 0}^{\tilde{\mathbb{P}}}[\cdot]$, we get the inequality

$$\mathbb{E}_t^{\tilde{\mathbb{P}}} [M_{s \wedge \tau_a}^\pi] \leq M_t^\pi + \mathbb{E}_t^{\tilde{\mathbb{P}}} [\mathcal{M}_{s \wedge \tau_a}^c + \mathcal{M}_{s \wedge \tau_a}^d],$$

with equality if $\pi = \tilde{\pi}$. Similarly to the arguments leading to (97), we have that $\mathbb{E}_t^{\tilde{\mathbb{P}}} [\mathcal{M}_{s \wedge \tau_a}^c] = 0$. To deal with \mathcal{M}^d , note that since $\underline{w}, \bar{w} \in C([0, T] \times \tilde{\Delta}_{N-1})$ and $\{\mathcal{U}_s\}_{t \leq s \leq T}$ is uniformly bounded (due to the fact that $-\tilde{\eta}$ is concave), we have that the integrand of the second integral in (103) is uniformly bounded and, thus, $\mathbb{E}_t^{\tilde{\mathbb{P}}} [\mathcal{M}_{s \wedge \tau_a}^d] = 0$ as well. The two previous facts, together with the initial conditions $H_t = 0$ and $\tilde{p}_t = \tilde{p}$, lead to

$$\mathbb{E}_t [M_{T \wedge \tau_a}^\pi] \leq M_t^\pi = e^{w(t, \tilde{p}_t, H_t)} = e^{w(t, \tilde{p}, 0)} = e^{\bar{w}(t, \tilde{p})}, \quad (105)$$

with equality if $\pi = \tilde{\pi}$.

(ii) The rest of the proof is similar to the post default case. From (102) and the Dominated Convergence Theorem,

$$\lim_{a \rightarrow 0} \mathbb{E}_t^{\tilde{\mathbb{P}}} [M_{T \wedge \tau_a}^{\tilde{\pi}}] = \mathbb{E}_t^{\tilde{\mathbb{P}}} [M_T^{\tilde{\pi}}], \quad (106)$$

From the previous limit and the fact that we have equality in (105) when $\pi = \tilde{\pi}$,

$$e^{\bar{w}(t, \tilde{p})} = \lim_{a \rightarrow 0} \mathbb{E}_t^{\tilde{\mathbb{P}}} [M_{T \wedge \tau_a}^{\tilde{\pi}}] = \mathbb{E}_t^{\tilde{\mathbb{P}}} [M_T^{\tilde{\pi}}] = \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[e^{-\gamma \int_t^T \tilde{\eta}(u, \tilde{p}_u, H_u, \tilde{\pi}_u) du} e^{w(T, \tilde{p}_T, H_T)} \right] = \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[e^{-\gamma \int_t^T \tilde{\eta}(u, \tilde{p}_u, H_u, \tilde{\pi}_u) du} \right], \quad (107)$$

since $w(T, \tilde{p}_T, H_T) := (1 - H_T)\bar{w}(T, \tilde{p}_T) + H_T\underline{w}(T, \tilde{p}_T) \equiv 0$. Similarly, from dominated convergence theorem, for every feedback control $\pi_s = \pi(s, \tilde{p}_s, H_s) \in \mathcal{A}(t, T; \tilde{p}, 0)$,

$$\mathbb{E}_t^{\tilde{\mathbb{P}}} \left[e^{-\gamma \int_t^T \tilde{\eta}(u, \tilde{p}_u, H_u, \pi_u) du} \right] = \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[\lim_{a \rightarrow 0} M_{T \wedge \tau_a}^\pi \right] = \lim_{a \rightarrow 0} \mathbb{E}_t^{\tilde{\mathbb{P}}} [M_{T \wedge \tau_a}^\pi] \leq M_t^\pi = e^{\bar{w}(t, \tilde{p})} = \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[e^{-\gamma \int_t^T \tilde{\eta}(u, \tilde{p}_u, H_u, \tilde{\pi}_u) du} \right],$$

where the inequality in the above equation follows from (105) and the last equality above follows from (107). This proves the assertions (1) and (2). \square

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